Projective geometry in the primary school curriculum : children's spatial-perceptual abilities.

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PROJECTIVE GEOMETRY IN THE PRIMARY SCHOOL CURRICULUM : CHILDREN'S SPATIAL-PERCEPTUAL ABILITIES

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ABSTRACT

Some fundamental questions are posed concerning the teaching and learning of geometry in primary schools. How can geometry be made more interesting, vital and relevant; and how can it be related to other subject areas? How may critical learning experiences be identified which can provide a conceptual framework and an intellectual clarity within a primary school curriculum?

Specific aspects of projective geometry are formulated to resolve the problem of disparate emphases in topological and euclidean approaches, thereby linking them both logically and psychologically; and enabling the learning of key concepts to be integrated with other subjects such as geography and art.

Implications for teaching are considered, especially those inherent in the historical development of geometry, the relationship between projective geometry and perspective, the influence of the secondary school curriculum on primary geometry, and the development of children's perceptions through projective experiences.

The study investigates ways in which children interpret and use line drawings. In addition, it is shown that children have strong preferences for projective rather than euclidean transformations.

A description is given of a projective geometry curriculum together with an account of its application in a primary school where it was evaluated in a variety of ways. Modes of geometrical thinking are suggested, the significances of these for teachers and learners are considered, and possible avenues for further research are discussed.

ACKNOWLEDGEMENT

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CHAPTER ONE

Introduction

An Examination of Problems in Primary School Geometry

It is the opinion of this investigator that much of primary school geometry today is lacking in intellectual and conceptual clarity. There is no integrated framework and no overall perceivable structure, that is, no available system of interconnected relationships between the various elements of the subject. Many parts are relevant and interesting as isolated topics, but do not provide an integrated and vital conceptual background for the child which is necessary if meaningful learning is to occur.

This lack of clarity may be formulated by listing the following fundamental questions: (a) How can geometry in the primary school be made more interesting, more vital, more relevant, more related to other subject areas, and thus more worthy of study by the child? (b) Why is geometry taught in schools so disjointed and unsystematic and how can the critical geometrical learning experiences be identified which will provide a cognitive and conceptual framework impinging more directly with the child's immediate environment? (c) Above all, within the available resources, how can geometry be given the necessary clarity without a big increase in time allocation in what is often considered an overburdened curriculum?

The Main Theme of this Study

This study attempts to resolve these issues by organising geometry into well integrated themes with a perceived structure which can take its place in the primary school curriculum. By presenting selected aspects of projective geometry provision is made for the development of a unique feature in the child's spatial-perceptual understandings affecting and influencing other aspects of their education.

The means may thus be supplied by which the relationships inherent within geometry can be perceived by the child and cognitive structures organised. A structured study of geometry may provide key concepts, (integrating related concepts in other subjects such as geography, science, and art) enabling fertile understandings in unexpected contexts and situations to develop.

A Definition of Projective Geometry

What is projective geometry and its relationship with primary geometry? For a mathematician, projective geometry is an often neglected part of classical geometry, popular in the past but of only marginal interest at the present time. In the context of the primary school, projective geometry may be defined as the study of perspective and line drawings, that is, two dimensional representations of three dimensional objects which includes the properties of straight lines and figures formed from them.

A Confusion in Present Day Primary Geometry

Where it is taught seriously, geometry in the primary school has two distinct emphases - namely euclidean and topological. The euclidean approach may be briefly described as the "usual geometry of shape and size". It deals with, for example, lengths and areas, angles and rotations, symmetry and reflections, shapes and scale models, properties of straight lines and associated figures. It is here that children often acquire skills with ruler and compasses, constructing triangles, measuring angles, fitting tessellations, and making elementary scale drawings.

Euclidean geometry is a subset of a wider geometry called topology. Topology is the study of continuous lines, of order along a connected line, and spatial relationships which are not necessarily as specific as those in euclidean geometry. Thus a topological vocabulary includes such words as inside, near, joined to, and between.

It is the investigator's belief that teachers perceive little connection between these two aspects of geometry and that they, and consequently the children, fail to see that euclidean concepts are contained within wider topological ones; that projective geometrical experiences can be used to link these aspects together both logically and psychologically. The topological, projective and euclidean approaches then proceed inter-dependently as a unified experience.

Increasing Geometric Relevance across the Curriculum

Primary geometry, as it is usually presented may often appear to be both irrelevant and confusing. Why should it be important, it may be asked, to learn such things as constructing triangles and measuring their angles, deciding whether something is inside a boundary or not, finding tessellation patterns or constructing circles? With some justification, such exercises may not appear to impinge on the everyday life of children. Indeed some teachers may teach geometry mainly to give some relief in the monotonous diet of sums and yet more sums.

Teachers, with some care and thought, may make these exercises more relevant and interesting. For example, the study of perspective in line drawings can lead, on the one hand, to a consideration of views from above and in front, and hence to plans, maps, and elevations or on the other to an increased understanding of sketches, drawings, and photographs. Such experiences and learnings taught systematically within a partly hierarchical geometry then becomes an integral part of a perception of the environment contributing a distinctive element to children's cognitive maps.

A Projective Geometry Curriculum

The investigation and experimental work of this study is the production and evaluation of a programme of units in projective geometry which may be integrated into a primary school curriculum. However, in order to proceed it is first necessary to seek some answers to a number of further questions in addition to those posed earlier in this chapter. These are: What are the teaching implications inherent in the historical development of geometry, especially non euclidean geometry? What is the relation between projective geometry and perspective? What do studies in the development of perception contribute to primary geometry? How does projective geometry relate to geography, science and art in the primary school? How does the child interpret line drawings within the child's existing cognitive structures? Do children prefer projective rather than other geometric transformations? What influences should the secondary school curriculum have on primary geometry? How does this investigation compare with other related research?

Outline of the Thesis

After the introductory chapter the second considers geometry in education. The aspects reviewed are the nature and historical development of geometry; an assessment of the interrelationship between primary and secondary school geometry as mathematical approaches to the subject itself change; a review of recent research in the area on two broad fronts, where the work of Piaget and Inhelder is studied in relation to other contributors supporting and contradicting their conclusions; relevant research in geography, art, science and perception; and finally the chapter concludes with theoretical and practical considerations of developing a curriculum in projective geometry.

Chapter three investigates line drawings and children's interpretations of projective geometrical ideas and perspective; children's preferences for projective and other geometric transformations are then researched; with a conclusion that children have a strong preference for projective transformations and that this has classroom implications.

Chapter four deals with the implementation of the study. As there were few precedents or directly relevant guidelines, the investigational procedures were necessarily very tentative. Firstly, pilot tests were constructed to try to ascertain appropriateness for age groups. Secondly the main research followed. This could not be designed to evolve along the lines of the pure classical evaluation experiment. Thirdly, questions arose which required further study of other research findings and modifications to the curriculum were built in to the development of a conceptual structural approach as the research proceeded. Evaluation processes included pre-test post-test assessments, reflection on teaching experiences and discussion with teachers and children.

Chapter five discusses implications derived from the research. Modes of geometrical thinking are hypothesised and the significance for teachers and learners considered. A curriculum package is examined and possible avenues for further research in the area discussed.

CHAPTER TWO

Geometry in Education

Geometry : Its Nature and Historical Development

Geometry as useful. Mathematicians have differing ideas on the nature of geometry and its relation to other branches of mathematics. At its most elementary level it may be considered as the exploration of space. This is the "useful" geometry needed by travellers, surveyors, architects, builders, map-makers, and others. Dienes and Golding (1967) make this point as does Sawyer (1977). The ancient origins of "useful" geometry are well documented. The early Egyptians are conventionally put into the category of those concerned with applied geometry.

Geometry as an abstraction. Geometry can be studied in purely abstract terms within the mind, formulating and projecting its own internal logical properties, rules, and procedures without relevance to anything else. Geometry then becomes the study of position, location and later is the study of the properties of geometrical figures. Copeland (1974), Courant and Robins (1961), Euclid (Heath 1956), and Willson (1977) make this clear in their respective books - though obviously from very different viewpoints. Geometry becomes a picture in the mind, a picture drawn from the real physical world but is not a picture of that real world. A degree of abstraction is realised. A view exists that the true essence of geometrical ideas has an independence from the real life situation from which it has been developed. Geometry is then a set of mental pictures and the study of the relationships that exist between different entities in these pictures.

Geometry as a logical structure. This viewpoint is expressed in the analyses of Coxeter (1969), Bishop (1980), Hardy (1925), Euclid (Heath 1956), Choquet (1969), Kline (1972) and in the N.F.E.R. Visualising Test "Q". The basic proposition is that geometrical ideas can better be considered as a structured collection of logical and mathematical structures. These structures become geometry or separate geometries according to the ways in which they are organised. These organisations are directed by topological, projective and various metric laws. (Piaget and Inhelder, 1956; Bell, 1976; Gans, 1973; and Burn, 1975). They are implicit in Nuffield Foundation Teaching Project publications, Shape and Size (1967) and Environmental Geometry (1969). Thus a common strand of geometry as a logical structure may be identified in books which start from very different bases of understanding and with different purposes in mind.

Geometry as a study of relationships. Piaget and Inhelder (1956) suggest that the child sees geometrical relationships by classifying experiences and by ordering activities. The relationships could, for example, involve correspondences and be dichotomic, asymmetric or ordered. Geometric relations are used by Escher (1967) and are commented upon by Vernon (1962) and Bruner (1960). Further relationships are proximity, separation, enclosure, continuity, straightness, parallelism shape, proportion, size, length, distance, and others. (Cox, 1977; Dienes, 1966b; Kagan & Lemkin, 1961). Authors generally agree that the child is led, by a slow process, often with faulty thought processes and misconceptions towards some form of geometrical thinking. (Beard, 1957; Skemp, 1963; Vygotsky, 1962; Baker, 1924).

Geometry as exhibited by behavioural changes. Gagne (1970) makes the point that geometrical thinking is demonstrated through behavioural changes and others, Russell (1973), Cassirer (1957), and Bruner (1960) assert that this has transfer value in dealing with non geometrical situations. Macfarlane Smith (1964), Vernon (1962) and Hebb (1949) make the point that the child's geometrical or spatial abilities may be capable of measurement. These geometrical abilities, whether actually measured or not, may be used on occasions to solve a variety of problems - some of these problems may be of a mainly geometrical nature but some may not. (Bell, 1976; Dienes, 1966b; Matthews and Matthews, 1978).

A further consideration is the use and role of language and verbalisation and how these may help or hinder in the problem solving exercise. Thompson (1959), Gattegno (1971), Jones (1912), Bruner (1960), Steffe and Smock (1975) provide a variety of analyses in this area.

Geometry as a process. Geometrical learning may be considered as a process leading to the formation of cognitive structures which contain specific sets of laws and relationships with particular modes of thinking. This process, to be successful, leads from an accumulation of mental pictures to an interlocking network of concepts, (called cognitive maps by some authors) with an associated language for communication which has the capacity to enrich the child's life. (Freudenthal, 1971; Choat, 1975; Bell, 1976; Wilmore, 1970).

A great name in the history of geometry : Euclid. Euclid systematised geometrical thinking in his Elements which consisted of thirteen books. It is likely that there were more which have been lost. They were all lost to the West until rediscovered via the Arabs in the Middle Ages. The scope and approach in Euclid's compilation was so comprehensive that it appeared unlikely that any further significant advances could be made. Euclid was geometry. In 1482 the first printed edition of his work was published and was, for many years, the only geometry studied and taught.

The situation is very different today. These developments are set out in a history of geometrical ideas as classified by Smith (1923) into four periods: (a) The synthetic geometry of the Greeks, including Euclid's work, work by Appolonius and others. The names of Pythagoras, Archimedes, Thales are associated with this period. (b) The birth of analytic geometry. The work of Desargues, Kepler and others merged into the co-ordinate geometry of Descartes and Fermat. (c) The application of calculus to geometry. (d) The renaissance of pure geometry including the projective geometry of Poncelet, the work of Steiner, the non euclidean hypotheses of Lobachevsky, Bolyai, Gauss, and Rieman, the foundations of geometry by Hilbert.

This is, of course, an oversimplification, for example some ideas on co-ordinates were used before the time of Descartes and Euclid may have had some intuitive ideas that his work was not the completed story of geometry.

Euclid's postulates : The fifth postulate. Little is known about the life of Euclid. This is rather surprising as his treatise overshadowed all other work in Geometry. He was the great compiler. He took the results of other mathematicians and treated them in a systematic manner. In particular he took the ideas of Aristotle and applied them to geometry. This proved to be his greatest achievement and although some geometrical work was developed in later classical times by mathematicians such as Eudoxus, Ptolemy, Menelaus and Pappus such was his eminence that it was some two thousand years or so before any major shifts occurred.

Euclid used a series of axioms, postulates and definitions as ingredients of a logical hierarchy. The definitions used ideal points, lines, circles and so on to set up the building blocks of the system. The axioms, of which there are nine, were assumptions which are self-evident truths called common notions. These are not geometrical in nature. An example is "the whole is greater than the part". The postulates were assumptions which are geometrical in nature and are listed as follows.

Let the following be postulated: (a) To draw a straight line from any point to any point. (b) To produce a finite straight line continuously in a straight line. (c) To describe a circle with any centre and any length of radius. (d) That all right angles are equal to one another. (e) That, if a straight line falling on two straight lines (in the same plane) makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.

The rather distinctive nature of the last one, Euclid's fifth postulate, has excited the interest of mathematicians through the ages. Consequences of the fifth postulate include, with some results about parallel lines, the theorem of Pythagoras and the uniqueness of a circle through three non collinear points. The fifth postulate hence has important consequences in geometry and is a special postulate being rather different from the others. Attempts have been made to prove the fifth postulate from the other four. Proofs which have appeared have been erroneous and alternative postulates were put forward by others to replace the fifth postulate by another not so distinctive one. It now seems possible that Euclid knew of some of the difficulties in this postulate and wrote it in this form so that its distinctive nature was made more apparent.

Some of the substitutes for the fifth postulate are listed below: (a) Through a point not on a line there passes not more than one parallel to the line. (Playfair's Axiom). (b) Two lines that are parallel to the same line are parallel to each other. (c) A line that meets one of two parallels meets the other. (d) If two parallels are cut by a transversal the alternate interior angles are equal. (e) There exists at least one triangle with an angle sum of two right angles. (f) For parallel lines all perpendicular distances from either line to the other are equal. (g) Similar triangles exist which are not congruent.

Mathematicians who have worked on the postulates include: Zeno (1st century B.C.); Posidonius (1st century B.C.); Ptolemy (2nd century A.D.);

Proclus (c. 460); Al-Tusi (c. 1200); Wallis (c. 1600); Saccheri (c. 1773); Lambert (c. 1766); Legendre (c. 1794); Playfair (c. 1795). The work of these and others led to the rapid development of pure geometry in the nineteenth century.

It is interesting that Dubbey (1970) gives Euclid's postulate in the form of Playfair's Axiom. He makes amends, however, by pointing out the special nature of the second postulate also, as both the second and the fifth postulate are concerned with the indefinite extension of a line and cannot for this reason be verified empirically.

Other early geometrical developments. The relationship of astronomy to mathematics was the incentive for much study in the ancient world. Hawkins (1970) makes out a convincing case for considering Stonehenge and other sites as astronomical observatories and is supported by Thom (1967) and later by Wood (1978) who both see geometrical implications in the siting of relationships between the stone structures. Thus there appears to be a parallel synthetic geometry associated with lines of sight in astronomy.

Renaissance art. In a very different field, attempts were made during the Renaissance to depict a real world rather than use symbolism. An attempt was made to introduce perspective into drawings and paintings by Italians such as Brunelleschi (1377-1446), Uccello (1397-1475), Alberti (1404-1472). Alberti imagined rays of light coming from the scene to the eye and hence ideas were developed about projections and sections. Durer (1471-1528) went much further mathematically. Leonardo da Vinci (1452-1519) adapted these ideas in the fields of art and science adding his own unique contribution to the attempt to depict the real three dimensional world on a two dimensional canvas.

Non euclidean geometry. The next major development came from Girard Desargues (1591-1661) generally recognised as the forerunner of projective geometry.

The Renaissance of pure geometry came largely as a reaction against the analytic geometry which held sway. Carnot (1755-1823) wished to free geometry from the hieroglyphics of analysis and Study (1862-1922) called the algebraic processes of co-ordinate geometry "the clatter of the co-ordinate mill". Monge (1746-1818) showed how to project a three-dimensional object onto a horizontal and a vertical plane and Poncelet (1788-1867) then developed some projective ideas and was the first mathematician to realise that projective geometry was a new branch of mathematics.

Thus began the notion that geometry was not merely the study of Euclid's Elements but a base which could be expanded into new branches. Many elegant and interesting theorems were developed in the new study of projective geometry. Some of these will be mentioned later. Gauss, Lobatchevesky and Bolyai realised that the euclidean fifth postulate could not be proved from the others and the axioms, and that the fifth postulate or an equivalent statement was needed for euclidean geometry. Lobatchevsky rejected the postulate and replaced it by the following (a figure at last).

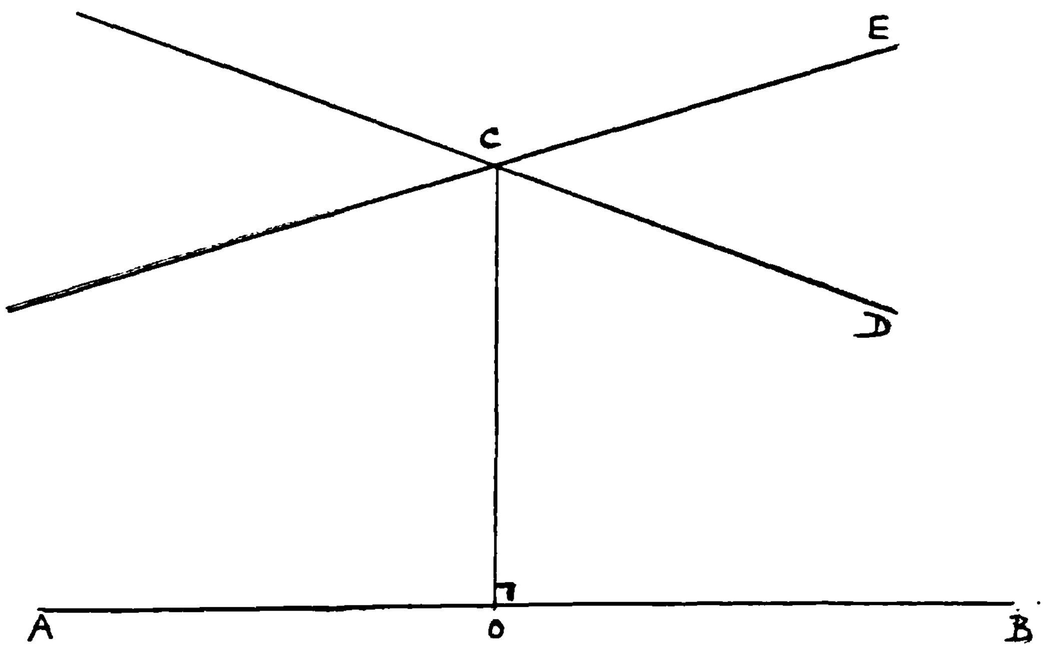


Figure 1. Lobatchevsky's Postulate.

Let AB be a line and let C be a point not on AB then a line through C falls into one of two sets, namely the set of lines which meet AB and the set of lines which do not. The set of lines which do not meet AB have two lines CD and CE which are the boundary between the two sets. CD and CE are said to be parallel to AB and accordingly any line through C between CD and CE is one of the set not meeting AB and is, therefore, _a_ parallel line to AB.

Although Lobatchevsky did not prove that this new set of axioms and postulates was self consistent and contained no internal self contradiction, he nevertheless believed that this was the case and on the strength of this belief can be credited with producing a non-euclidean geometry. Bolyai came to the same conclusion at about the same time and both obtained some of their ideas from Gauss. This modification of the fifth postulate produces a geometry now called hyperbolic geometry. If, on the other hand, the fifth postulate is changed to suggest that _no_ lines can be drawn parallel to a given line then, again, a consistent geometry is produced. This was presented by Riemann (1826-1866) and is now called elliptic geometry.

The names elliptic, hyperbolic, and parabolic or euclidean were given by Klein. Elliptic geometry was also separated into double elliptic geometry and single elliptic geometry as detailed in Gans (1973) and Kline (1972). When it became accepted that there existed more than one geometry then the place of a particular geometry in the set of geometries needed to be considered.

It was Felix Klein (1849-1925) in the "Erlangen Program" who pointed out that different geometries were distinguished by the kinds of one to one transformations they allow. Then a geometry beaame the study of the complete sets of invariants under transformations allowed in the geometry. Thus a classification of geometries may be shown by the following figure.

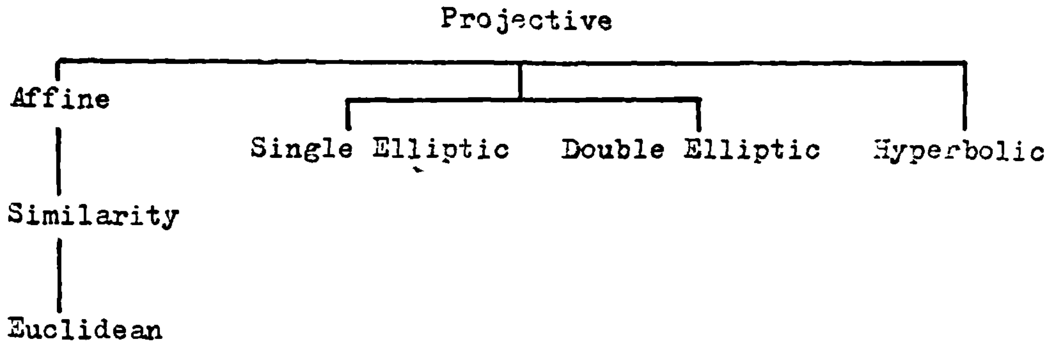


Figure 2. A classification of geometries.

The geometrical hierarchy of transformations may also be made more specific by stating the properties which are preserved in general transformations within those allowable in the geometry.

In a more detailed study of the development of non euclidean geometry it is interesting to note that the resurgence of significant creative activity in geometry lagged behind that in algebra. Apart from the creation of the mathematical system of perspective and the incidental geometrical work of the Renaissance artists, very little of consequence was done in geometry from the time of Pappus to about 1600. What was needed, and did arise to direct the minds of mathematicians into new channels, were new problems. One problem had already been raised by Alberti namely, what geometrical properties do two sections of the same projection of an actual figure have in common? A real scene is viewed by the eye regarded as a point. The lines of light from various points of the scene to the eye are said to constitute a projection. According to the system, the painting itself must contain a section of that projection, the section being mathematically what a plane passing through the projection would contain. Now suppose the eye at O looks at the horizontal rectangle from above as shown in figure 3, as

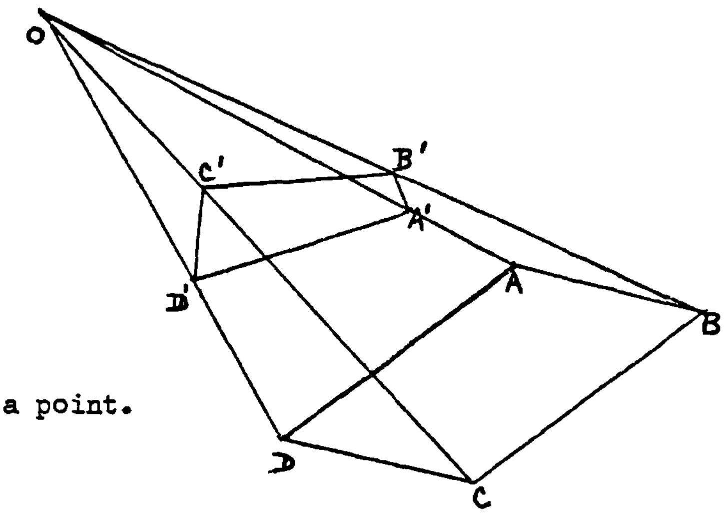


Figure 3. Projection from a point.

ABCD. The lines from O to the points on the four sides of this rectangle constitute a projection of which OA, OB, OC and OD are typical lines. If a plane is now interposed between the eye and the rectangle, the lines of the projection will cut through the plane and mark out on it the quadrangle A'B'C'D'. Since the section, namely A'B'C'D', creates the same impression on the eye as does the original rectangle, it is reasonable to ask, as Alberti did, what geometrical properties do the section and the original rectangle have in common? It is intuitively apparent that the original figure and the section will be neither congruent, nor similar; nor will they contain the same area. In fact the section need not be a rectangle.

There is an extension of this problem: Suppose two different sections of this same projection are made by two different planes that cut the projection at any angle. What properties would the two sections have in common? The problem may be further extended. Suppose a rectangle ABCD is viewed from two different locations O' and O". Then there are two projections, one determined by O' and the rectangle and the second determined by O" and the rectangle. If a section is made of each projection, then, in view of the fact that each section should have some geometrical properties in common with the rectangle, the two sections should have some common geometrical properties.

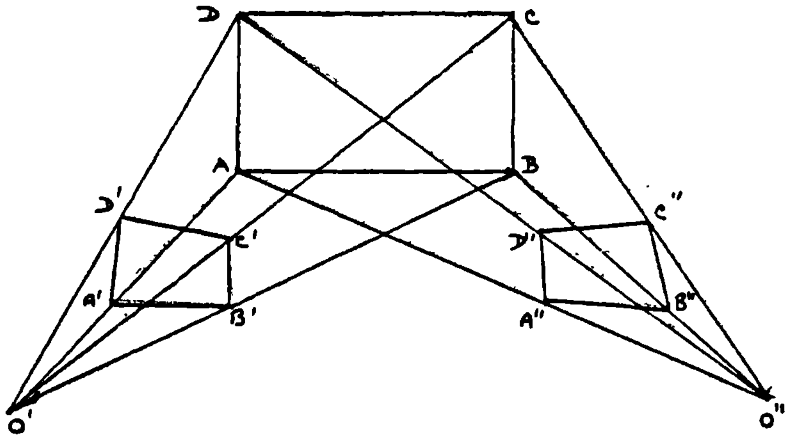


Figure 4. Projections from two points.

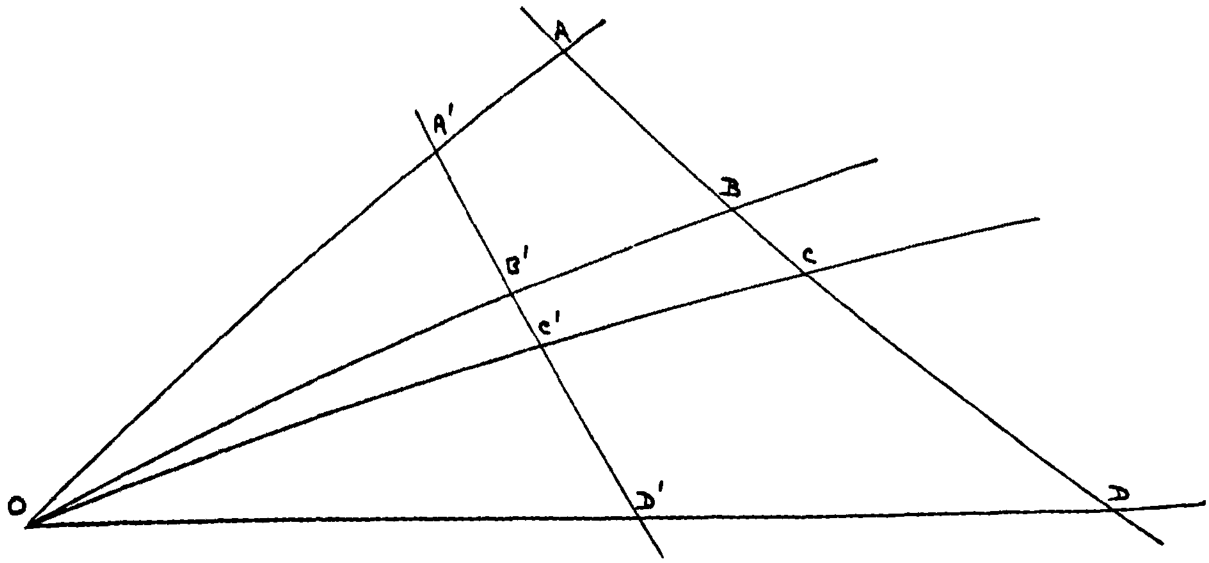
Some of the seventeenth-century geometers undertook to answer these questions. They viewed the methods and results they obtained as part of euclidean geometry. However, these methods and results, while indeed contributing much to that subject, proved to be the beginning of a new branch of geometry, which in the nineteenth century became known as projective geometry.

Desargues and others. The man who first took up directly the problems just sketched was the self-educated Girard Desargues, who was an army officer and later became an engineer and architect. Desargues knew the work of Applonius and thought that he could introduce new methods for proving theorems about conics. He did, and, indeed, was fully aware of the power of these methods. Desargues' main theorem on triangles and other theorems of his were published in 1648.

Poncelet was the first mathematician to appreciate fully that projective geometry was a new branch of mathematics with methods and goals of its own. Whereas the seventeenth-century projective geometers had dealt with specific problems, Poncelet entertained the general problem of seeking all properties of geometrical figures that were common to all sections of any projection of a figure, that is, remain unaltered by projection and section. This is the theme that he and his successors took up. Because distances and angles are altered by projection and section Poncelet selected and developed the theory of involution and of harmonic sets of points but not the concept of cross ratio. Monge had used parallel projection in his work; like Desargues, Pascal, Newton, and Lambert, Poncelet used central projection, that is, projection from a point. This concept Poncelet elevated into a method of approach to geometric problems. Poncelet also considered projective transformation from one space figure to another, of course in purely geometric form. This should not suggest that projective ideas began after 1000 A.D. Some fundamental results in projective geometry date from the time of the Greeks.

Projections and cross ratios. Pappus discovered the cross ratio property and, of course, his famous theorem. These tended to be isolated results standing outside the euclidean development and the framework of thought used in their description was euclidean.

The cross ratio property of projections is a property associated with distances between a set of four points and their transformations. A "ratio of ratios" is preserved as invariant thus in the following figure.



The ratio of ratios $\dfrac{\frac{AB}{BC}}{\frac{AD}{DC}} = \dfrac{\frac{A'B'}{B'C'}}{\frac{A'D'}{D'C'}}$ that is cross ratio is invariant under projective transformations.

Figure 5. Cross ratio property.

Later developments in projective geometry as an extension of euclidean geometry tend to be analytic and numerical and beyond that needed here for this study. The concept of cross ratio, for example, will not be part of the study of childrens' geometry up to the age of ten or so, involving as it does the concept of length which is a congruent invariant. Gans (1973) makes it clear that the phrase "Non euclidean" geometry is usually taken to refer to a study of those geometries mentioned previously.

The geometrical hierarchy of transformations may also be made more specific by stating the properties which are preserved in general transformations within those allowable in the geometry. The sets of allowable transformations in most cases form a group; that is the set is closed, associative, has an identity transformation and each transformation has an inverse.

The projective affine similarity and isometry groups can be classified according to their effects on lengths: (a) Isometry group, length in any direction is invariant. (b) Similarity group, length ratio in any direction is invariant. (c) Affine group, length ratio in a given direction is invariant. (d) Projective group, ratio of length ratio (cross ratio) in any direction is invariant with each subgroup possessing the invariant properties of the overall group. Thus cross ratio is preserved in (a), (b) and (c) as well as (d). An explanatory figure is given in appendix 2. Klein's classification of geometries put the geometries into perspective so to speak and suggested new developments.

Differences between Projective Geometry and Perspective. These differences need to be considered briefly. Generally the differences are not made clear or else appear to be overlooked altogether. They are mentioned here as possible avenues for further study but do not appear to be of crucial concern. Basically projective geometry preserves straight lines (and other invariants such as cross ratio) but it appears that perspective does not do so. This is perhaps best explained by an example.

Consider an observer in a static balloon directly above a straight railway track. Directly below the observer he will see a pair of parallel lines. Away towards the horizon he will see a pair of converging straight lines (according to projective transformations) and away towards the horizon in the opposite direction he will again see a pair of converging straight lines; yet the rails do not appear to have any sudden changes of direction. So the converging "straight" lines cannot be straight but must curve in some way. Hence, seeing straight lines in perspective must involve some curving of these lines and so perspective transformations, depending as they do on the curvature of the earth, are not projective transformations (which preserve straightness). Without the curvature of the earth the straight lines would not have a horizon upon which to converge. Thus projective transformations of scenes, preserving straight lines, are only approximations of the scene in true perspective - rather like a flat map only being an approximation to the curved surface of the earth it is representing. In fact, in the original example the railway lines were actually not straight but circles on the surface of the earth - and incidentally at the most one of them could be part of a great circle. Perspective and perception are inter-related. Willman (1966) observes that a perspective drawing is geometrically correct only if it is capable of giving rise to the same retinal pattern as the three dimensional scene it purports to represent. Even this appears, however, to be an oversimplification and in view of these problems and as the main concern of this study is geometrical; further considerations of perspective have been left. Projective transformations will be taken by the investigator as being sufficiently accurate for most purposes.

Topology and Hilbert. Topology is the study of properties invariant under one to one transformation continuous both ways. It began with problems such as the four colour problem and that of the famous Konigsberg Bridges and led into general topology, concerned with concepts such as connectedness, betweenness, and continuity. Hilbert supplied some axioms to betweenness which Euclid omitted in his list; an example is "to any two points A and B on a line there is at least one point C on the line AB such that B lies between A and C". These were needed as extra axioms to prevent illogicalities in geometrical proofs and can be seen against a background of increasing geometric awareness.

A simplified version of the hierarchy of groups of transformations, showing some of the invariants, is given. This has been modified from Mansfield and Harris, 1968 Year 2, Autumn Term, p. 82. (Appendix 2).

Primary and Secondary School Geometry : Changing Approaches

Difficulties in an analysis of school geometry. It is inherent in the nature of the task that it is difficult to generalise about geometrical education. Nearly every generalisation will have some limitations and exceptions. In spite of the constraining effects of the public examination system, geometric experiences given to children vary considerably. A major problem is that of the various interpretations which may be given to the meaning of the word geometry. For this reason questionnaires and personal dialogues are open to misinterpretations. Neither do school texts, by themselves, of necessity give a good indication of geometrical activities in the schools. The individual school, the individual teacher and the individual pupil, to varying degrees, have freedom of a sort to follow an individualistic route through or around geometrical activities. Some books for teachers sometimes give impressions of what teachers could be doing and others reflect what is generally happening in the schools. These are not good guides to general geometric expectations.

Teachers may also hide their deficiencies in their responses to questions about their geometrical teaching and even their own levels of attainments may leave much to be desired. It is only by sifting through as much evidence as possible that a picture of geometrical teaching begins to emerge. Thus it is valuable to consider commentaries by interested individuals or groups, suggestions for innovations, consideration of school texts, discussions with educators within and outside the schools, official and unofficial publications, school texts, work with children in schools, research data and a review of research in the area. From the historical survey it may be seen that geometry has been studied from early times - that it has generally been considered as worthy of study and has been seen as having a beauty and a logic which is quite distinctive and applications which are far from trivial.

<u>Euclidean geometry in the secondary school</u>. Geometry up to the 1950s is perhaps typified by C. V. Durrell (1925). Although the approaches in the earlier years of the century were considerably different nevertheless

the general aims and the specific content were unashamedly euclidean. The emphasis may well have been different in different classes but the euclidean nature of the content was fairly constant.

From the viewpoint of the present day it appears that everything up to the 1950s was so clear cut for the secondary teacher of geometry, it was taken for granted that children should learn geometry, that a geometrical education should begin at secondary school, that it should be taught in a way that would be approved by the Greeks, that logic should be the basis of learning, that geometrical constructions should be of limited concern used mainly to back up the logical approach, that it was not very respectable to use either arithmetic or algebra to help in geometry as somehow this made the subject less elegant and not so worthy. There was also a hint that the subject was at its best, at its purest, when it had no practical applications, that even the act of representing geometrical ideas by drawings somehow tainted the purity of geometry with a 'mundane' or 'wordly' contamination.

Comparison with classics as a discipline. The rationale for teaching secondary school geometry seemed to be the same as that for teaching the classics - perhaps in their different ways both geometry and the classics have withered and all but faded into insignificance. It is rather surprising that geometry should have been, and still is, on occasions, taught this way when the next step at Advanced level is so often co-ordinate geometry where the algebra and geometry are inextricably mixed. Yet even here approval tends to be given more for a purely geometric proof than an algebraic one "the clatter of the co-ordinate mill". Perhaps both pre and post O level geometry was taught that way because that was what was expected? As the teaching of the classics, with one or two minor modifications, fell into decline, other pressures have ensured that the teaching of geometry has not suffered the same fate. There are several reasons for this. Of paramount importance is the obvious application of geometrical ideas in industry and science and technology. Another reason may be the high regard in which mathematics appears to be held by society. There are pressures on children to succeed in this subject. There are pressures on teachers to go on to new work before the previous work is sufficiently well understood. There are conceptual difficulties in a sequential study and many children learn to dislike a subject in which they feel they cannot succeed. It is easy to make the subject difficult, arid, irrelevant, and distasteful. Nevertheless it is generally felt that mathematics is important, of value both to society and to the individual; together with a strong, but silent, agreement that mathematicians are the ones most likely to know the sort of mathematics that should be taught. Industrialists may, occasionally quite rightly, complain about standards, but they do not often complain about content. This is an area where they feel unsure of themselves. Another reason why geometry did not disappear from the curriculum was the pressure, most evident about the 1950s for unification, in Britain, of the different strands of mathematics - with arithmetical, algebraic and geometrical ideas being used whenever appropriate in any part of mathematics. This was reflected or produced by the O level examination papers which were no longer separated into arithmetical, algebraic and geometrical assessments. Without this silent revolution having taken place, almost by default, geometry would have been under considerably more pressure to survive in the secondary school, though the reasons for the unification were more honourable, more pressing and more justifiable than that of preserving the place of geometry in schools. Breaking down the mostly artificial barriers between the various subjects within mathematics was intended to make them more relevant and more readily acceptable to school children.

A period of transition : emphasis on the linear map. In the 1960s and 1970s the introduction of so called modern mathematics into schools and colleges had a major effect on the teaching of geometry. The approach to geometry was no longer euclidean and this in itself was a major concern to teachers with a euclidean background. This may be seen in the way that Courant and Robins (1965) develop the idea of geometry and geometries. They begin from a euclidean base with geometry defined as dealing with the properties of figures in the plane or in space. Then they build towards a transformational, Erlangen view by asking whether the concept of magnitude is essential to geometry. Geometrical figures may, they state, have even deeper properties which are not destroyed by transformations more drastic than rigid motions. A collection of theorems dealing with those deeper properties will be the geometry associated with the corresponding class of transformations.

The emphasis was placed much more on the concept of a linear map; that is, a linear transformation from a vector space to a vector space effected quite often by the use of a matrix to represent such a linear transformation.

Such an approach through linear maps emphasises affine geometry and omits topology and projective geometry, leaving a subset of the Erlangen programme with affine geometry as the most general geometry. Choquet (1969) suggests that children would still benefit from an approach to geometry based, like Euclid's, on concepts drawn from the physical world. But, at the same time, that the powerful and flexible tools of algebra should be at our disposal. He has affine geometry as his most general geometry, as he advocates the use of a vectorial approach. He avoids projective geometry altogether electing to present the concept of direction from the start. Choquet's modern setting thus tends to be algebraic rather than transformational.

A revealing comment which has implications in the whole level of approach to geometry is given by Budden and Wormell (1964) who state that the real issue is not between a recently achieved self-styled progress and a tradition, but between a syllabus based on algebra and one based on geometry. It seems to the authors that geometry had been relegated to being a branch of algebra.

It is the investigator's opinion that it is unfortunate that elegant results about circles and spheres, for example, tend to be ignored and projective geometry is relegated to a page or two of a school text and not related to the rest of geometry. Some texts also contain topological ideas, but again as isolated topics separated from the main stream linear map approach. "Will anyone dare to produce a (secondary) school course which includes theorems of people like Pappus, Desargues, Pascal, Brianchon, Ceva, Menelaus, Ptolemy, ... all could be understood and "discovered" (although not proved) by secondary school pupils. If we do not save something now from pure geometry soon, the day will come when no one knows (or cares) about some of the most elegant mathematics we have and mathematics will be a poorer subject as a result". Cornelius (1980) p. 19. Willson (1977) also suggests some experiments to bridge the gap between intuitive and formal topological notions.

Geometry in the secondary school has, on the whole, survived into the 1980s, but in a very different form from that which it had in the earlier years of the century (even though some schools hardly accept that changes have or should have taken place). The position of geometry in the secondary school is still a matter of debate with many different opinions and shades of opinions being advocated on its value.

Primary geometry. There is little tradition of teaching geometry in primary schools. Historically this is probably due to the emphasis on basic skills mainly in the three Rs. Not unnaturally, the efforts of teachers were concentrated in teaching to compute numbers and measures.

This was what was expected of them in the competitive 11+ selective examination. With the advent of comprehensive schooling some pressures were lifted and many went through a new phase of a liberated curriculum on Plowden lines. But old attitudes change slowly and generally work outside the three Rs is not accountable to the public in the same way. Nevertheless there was more freedom for teachers to select topics and, it should be noted, to avoid others. It appears that much of the work inspired and encouraged by interested parties such as H.M.I.s and advisers in such subjects as art, physical education, dance, drama, science, music and others has resulted in a climate of opinion in which it is reasonable to spend time and resources in these areas.

Primary teachers are, then, under pressure to broaden the educational base, possibly thereby, some would argue, weakening progress in basic subjects. The balance is fine and there is little time for frills. The primary curriculum is best seen as a number of areas of knowledge competing with each other for timetable space. Inevitably there are casualties. Primary teachers often attempt to find a solution by combining subjects under a topic heading. Thus geometrical ideas may be used to develop number concepts and spatial ideas only mentioned, perhaps, in art. One might expect geometry to be a feature of primary mathematics but there is only a fleeting mention of geometry in some text books, often at the end of a chapter. These may not be taken seriously by some teachers because of the difficulty of testing of geometrical concepts compared with the ease in which arithmetic algorithmic skills may be assessed.

The need for putting elementary ideas into the primary school was emphasised by the Nuffield Foundation Mathematics Teaching Project (1967). 'Shape' and 'Size' were used to deal with notions of similarity and congruence. It would have been useful to include other geometrical activities. I would have preferred "The world we live in, pictures and photographs, maps, shape and size". Though not so concise it would have given a better Erlangen and environmental view of geometry using topological, projective, and affine experiences as well as euclidean. This emphasis has been taken up by later primary texts which have firmly had a geometrical emphasis. Williams and Shuard (1970), Thyer and Maggs (1981), Glenn (1979) all establish geometry in the early years of schooling.

Ward (1979) in his review of what is taking place in primary school mathematics highlights some of the problems. No topological or projective topics are noted in his review and a bare mention of shears is made. There are, it may be seen, difficulties for the non specialist teacher to operate confidently within an Erlangen framework with an exclusively euclidean background.

This is not helped by the Department of Education and Science Handbook of Suggestions; Mathematics 5-11 (1979). This emphasises shape prior to considerations of a topological nature and is, I believe, a depressing document for anyone interested in the teaching of geometry.

There is an opposing view as expressed by Sauvy and Sauvy (1974). They set the topological scene for young children well. The development of the child's intelligence is considered and the usual topological invariants in two and three dimensions are discussed and introduced in some depth. The final chapter on point set topology seems quaintly out of place in such a book and is unlikely to be of much use to the practising teacher but there is a useful point which comments that the child's thinking may not be as straightforward as Piaget originally suggested. "However, after experiments carried out under the aegis of Piaget and by the Montreal research group, dealing in particular with the way in which children orient themselves and with the distinction they make between left and right, show that, from the age of four or five, projective space and euclidean space begin to be sketched in against a background of topological space . . . In conclusion, even if, after the age of two or three, children pass from sensory motor space towards a representational space, the latter is, in essence, no more than topological; the projective relationships (straight lines and perspective) and Euclidean relationships (metric) are present only in the form of a rough sketch" (p. 25).

Projective geometry in school : texts and suggestions. There is little explicit teaching of projective geometry in schools. A notable exception is a two page introduction in the School Mathematics Project, (1965-1970) but it appears that these pages are considered as optional by many teachers presumably because they are difficult to test.

In my limited experience as a teacher and lecturer these ideas are either ignored altogether or tackled as a topic quite independently of topological or affine ideas. It seems to follow that children obtain their ideas about projective geometry or perspective either in a very unstructured and haphazard way or not at all. On the whole the learning of projective geometries appears to be more or less neglected. The questions then arise as to what projective ideas children acquire, whether these develop during maturation, and whether ideas of projective geometry have a developmental aspect.

These questions can probably best be answered by experimental procedures designed to monitor children's activities and an examination of answers to questions which entail a projective viewpoint. It is this investigator's belief that projective geometry has the content, materials, activities, and ideas which can be contained in a conceptual framework which will make it a fascinating subject for study by all children.

Unfortunately the climate of opinion may well be against attempting experiences in projective geometry, at least until the later years of schooling. The Conference on the Teaching of Geometry at Carbondale, Illinois in March 1970 relegated projective geometry to older children. (Kaufman, 1971.) Furthermore it was suggested that long range objectives for a geometry programme for the 5 to 19 year olds present numerous problems. A rather lengthy list of topics was drawn up as follows: (a) Introduction of geometry in the early grades: Topological aspects, spatial geometry before plane geometry, experimental attitude towards geometry; motion geometry; symmetries. (b) Geometry for intermediate grades: Role of affine geometry; transition from an experimental to a theoretical - deductive attitude, the role of mathematicatian, order and orientation; concept of angle; geometry and number (algebraization); group theoretical aspects of geometry; congruence; similarity; measures such as length, angle measure, area, volume.

Projective ideas are briefly considered in Nuffield Foundation Mathematics Teaching Project (1969). The use of these ideas is quite attractive but is left in a rather inconclusive state. Affine transformations are hinted at with a figure containing a brick and congruent straw and paper skeleton models. But again this is given as a suitable activity for children without the mention of the underlying structural ideas.

There have been other attempts to consider projective activities for children. One of the most notable is by Dienes and Golding, (1967). The main emphasis is on a set of activities given on work cards. The learning of mathematical structures through apparatus is suggested. It contains, for example, a photograph of a child holding a rectangular lamina in such a way that the shadow fits on a rhombus. The basic pattern used is the Erlangen programme of transformational geometry turned into activities for children. In spite of their emphasis on mathematical structures they do not, in their series of cards, follow a hierarchical structural approach. The cards suggest the use of sun shadows (leading to affine and congruence transformations) before the use of point light sources (leading to projective and similarity transformations). There is, moreover, only one card exclusively on point light shadows. It may be argued that it is easier to use the sun than a lamp. A lamp tends to produce a rather fuzzy edged shadow whereas the sun does shine from time to time giving sharp and distinct shadows. It may be that the practical considerations involved in doing the experiments justify a change in the Erlangen order.

A different approach, yet again, has been taken by Bishop, (1980). He considers problems in perspective and in affine mappings of three dimensional figures into two dimensions. A mixture of these two may give rise to some interesting work with children. Finally, Copeland (1974) has a chapter headed Projective Geometry. He starts with a brief introduction to the Erlangen Programme explaining that in projective geometry, an object or idea such as a straight line is not considered by itself or in isolation, but in relation to how it looks from a particular point of view. Linear perspective (straightness) is then considered in relation to perception and reconstruction. Piaget's telephone pole experiment is described, emphasis given to straightness as obtained by line of sight, and shadow projections suggested.

A Review of Research in Learning Geometry with an Emphasis on Projective Ideas

Piaget, Inhelder, and Szeminska. A factor in the teaching of geometry which is slowly having an effect is the research by mathematical educationalists into how children learn geometry. It is being increasingly realised that such teaching and learning is not easy and that many factors have to be taken into account such as levels of maturation, previous learnings, ordering and sequencing of content, pacing, reinforcement, relationships of concrete materials to abstractions, variety of instances to ensure generalisation and so on. The hierarchical nature of much of geometry may entail a cumulative sequencing of learning experiences which leads to concept formation with frequent assessments to check progress. Piaget (1969), Piaget and Inhelder (1956), and Piaget, Inhelder, and Szeminska (1960) have had a considerable influence on the teaching and learning of geometries. Confusion can exist because of their use of the words "geometry" and "space". "Geometry" to them is basically euclidean with "space" being a more general term.

A general understanding of topological ideas and an attempt to get to grips verbally with an Erlangen accommodation are evident in Piaget and Inhelder (1956) where it is suggested that primitive, topological space is purely internal to the particular figure whose intrinsic properties it expresses. Moreover the only relation between two or more figures topologically is that of simple one-one bicontinuous correspondence. They are not without critics who believe that their ideas were sometimes confused. A major concern was that of explaining the Erlangen programme to readers operating within a euclidean paradigm.

General spatial ability. Piaget and Inhelder (1956) suggest that spatial activities may be of several forms of which three are (a) topological, (b) projective, and (c) transitional from projective to euclidean.

> "The teaching of geometry could hardly fail to profit from keeping to the natural pattern of development of geometrical thought, especially as the process is, to our way of thinking, in much closer conformity with the logic of mathematical construction than are more of the so-called "elementary" textbooks." (p. vii).

They suggest five aspects of spatiality, namely: (a) general, (b) pictorial through spontaneous drawing, and (c) pictorial through copying figures, (d) perspective through apparent shapes and shadows, and (e) perspective co-ordination. Essentially these aspects have three stages: (a) topological, (b) projective and euclidean, including topological, and (c) operational. The model has variations depending on the aspect under consideration but the following figure 6 is a reasonable fit to their categories.

According to Piaget and Inhelder, the child's conception of space develops through various stages from perceptual space through representational space to operational space. Perceptual space is the knowledge of objects which comes from direct contact with them. This contact may be visual or tactile or both. This spatial construction requires the presence of the object under consideration. A transformation is effected, mapping the object itself to its image in the child's perceptual space. This mapping may be of different forms depending on the child's previous experiences and learnings. Initially the visual and tactile transformations are not related, but gradually become more so as elementary topological concepts such as proximity, separation, order (spatial succession), enclosure and continuity are successively encountered.

The child's understanding of perceptual space is changing as further visual and tactile experiences are met leading to a greater relationship between the two and greater awareness of the topological constructions although projective and euclidean constructions (straightness, shape or size constancy) are not yet perceived. The child then develops an understanding of representational space. An object may now be considered in its absence and projective transformations may be made. The object is considered to be the "same" when seen, or recalled, from different viewpoints. Intuitive ideas of shape and length develop though there will be topological constructions and recourse to direct contact with the objects or figures being considered. The development is not as orderly as this account may suggest. Nevertheless, the mental recall of

(a) General

Perceptual	Representational	Operational Co-ordination
Objects visually, tactually considered.	Objects recalled in their absence.	Objects displaced relative to each other.
Shapes perceived topologically (proximity, separation, order, enclosure, continuity).	Shapes imagined projectively (object constancy) and euclideanly (shape and size constancy) also.	Shapes explored systematically reversable, mental manipulation effected.

(Perceptual space) $\xleftrightarrow[\text{interaction}]{\text{conflict}}$ (Representational space) $\xleftrightarrow[\text{interaction}]{\text{conflict}}$ (Operational space)

(b) Pictorial through spontaneous drawing

Synthetic Incapacity	Intellectual Realism	Visual Realism
Mainly egocentric and topological.	Developing projective and euclidean aspects.	Perspective, proportions, length aspects develop interdependently and operationally.

(c) Pictorial through copying figures

Stages O and I	Stage II	Stages III and IV
O Pure scribble.	IB to IIA Straightness.	III Further euclidean relations, reversibility flexibility are partial in IIIA, complete in IIIB.
IA Open and closed distinguished.	IIA Length and distance.	
IB Other topological relations.	IIB Elementary euclidean relations.	

(d) Perspective through apparent shape and shadows

I None or little.	IIA Slight attempts to portray perspective.	IIIA Need for receding lines to converge seen but not organised.
	IIB Intuitive ideas more apparent, representation tends to be without perspective.	IIIB Visual realism tackled more systematically, perspective co-ordinated
		IV More complex situations mastered.

(e) Perspective co-ordination

I Little or none.	IIA Reproduction of egocentric, own point of view.	III Relativity of perspectives genuine.
		IIIA but incomplete.
	IIB Slight attempts to distinguish between different viewpoints.	IIIB now completed.

Figure 6. A schematic outline of the spatial ideas of Piaget and Inhelder (1956).

an object when no longer present takes into account in gradually increasing ways; object constancy, (different projective views of an object are recognised as being of the same object); shape constancy, (different similar views of an object are recognised); and size constancy.

A later psychological stage was also suggested. In this stage objects are considered in relation to one other. If a displacement takes place (whether visually, tactually or mentally) the child is able (with increasing ability) to reverse this displacement to give the initial configuration. This leads to systematic exploration and to reversible mental manipulations. This stage is denoted by the authors as operational co-ordination.

Perceptual space then leads into representational space as contradictions between the two become apparent and are resolved by interaction. Then representational space leads into systematic, reversible or operational space.

Pictorial space, spontaneous and copying exercises. Piaget and Inhelder next considered pictorial space (the construction of an image). They investigated Spontaneous Drawing and then the Copying of Specific Figures ("the drawing of geometrical figures"). They used Luquet's three stages of spontaneous drawing, viz. synthetic incapacity, intellectual realism and visual realism.

In the first stage of synthetic incapacity, the child attempts to show topological relationships. If the figure is complicated some of the relationships may break down. To deal with proximity, for example, the child may possibly ignore aspects of continuity or order. Projective and euclidean relationships are lacking and only begin to develop in the second stage, intellectual realism. Logical constructs are used to show hidden but actual details - transparency or a multiplicity of irreconcilable points of view.

About the age of 8 or 9 the type of spontaneous drawing takes on visual realism. Perspective, proportions, lengths are taken into account in various ways, though these not only develop slowly but also surprisingly side by side interdependently rather than one preceding another.

It is in the area of the copying of specific figures that Piaget & Inhelder's ideas are open to criticism. Four stages of development, zero to three are put forward. Stages 1, 2 and 3 reflect in copying figures the perceptual, representation and operational aspects of the previous chapter.

(a) Stage zero is pure scribble with little attempt at copying. (b) Stage One is mainly topological (corresponding largely with perceptual space). IA open and closed curves tend to be distinguished. IB other topological relationships are considered and to varying degrees the drawings show proximity, separation, order, enclosure, continuity and so on. (c) Between Stage One and Stage Two attempts are made to distinguish straightness from non straightness - an essential projective aspect. (d) Stage Two is increasingly euclidean (corresponding with representational space). IIA length and distance are considered. IIB elementary euclidean differences are increasingly distinguished. (e) Stage Three is increasingly "operational" (corresponding to operational space), that is, the relationships are dealt with more abstractly and flexibly with mental reversibility as specified in the previous chapter in operational space.

In Stage IIIA these interdependent approaches are only seen partially but are complete in Stage IIIB when the exploration is systematic, organised and reversible. "It will be noticed that some of the models emphasise topological relationships while others are simple euclidean shapes" (p. 53). Here the implication is that shapes are separated into disjoint topological and euclidean sets. In spite of their understanding of the Erlangen programme the authors lapse here into a euclidean viewpoint - that is considering properties of figures rather than the transformations from a figure to its image. It is the transformation and not the figure itself which is topological or euclidean in nature. This point is developed in more detail in the account of the research of Martin (1976 a, b, c) later in the study.

Perspective ability. A nice example of the child's conception of space is provided by the straight line. Perceptually the child is able to distinguish straight from non straight towards the end of Stage I but does not have the ability to construct the straight line. This is a representational activity. In Stage IIA the straight line may be constructed parallel to the straight table edge but it is not until Stage IIB that consideration of multiple points of view gradually leads towards constructing a straight line oblique to the straight table edge. Straight lines may be imagined or constructed using (a) a projective approach - that of aiming or sighting along the line - this does not use a euclidean approach. Two euclidean approaches are (b) the shortest distance between two points and (c) the line which does not change direction (a line which maintains its shape when rotated about its own axis - formed by folding a piece of paper in two). These three approaches tend to be psychologically interdependent as has been seen operationally in Stage III previously.

Consideration of straightness leads to ideas of perspective - the power to imagine straight lines facing in any direction. Piaget and Inhelder used isolated objects, such as a needle, and by using a variety of viewpoints, considered perspective changes. Later, shadows of a figure using a lamp were also used. They concluded that there is little sign of perspective thought at the level of perceptual ability and that the representational level tends to be without perspective ideas (though increasingly slight attempts are made to take account of perspective). Only at the operational co-ordination, visual realism, stage is the need for perspective seen for example the need to made receding lines converge.

At Stage IIIB visual realism is tackled more systematically with perspective adequately co-ordinated.

A further more formal perspective activity is needed to deal with the added complications of the shadows of more complex objects, such as a single or double cone where a co-ordination of perspectives seems to be required.

Co-ordination of perspectives is examined at great length by Piaget and Inhelder (1956). Any broad generalisation must fail to do justice to the many fine points which they made. At the early stages children, they claimed, have a "synthetic incapacity" to co-ordinate perspectives. Later at Stage IIA they are only able to represent or reproduce their own point of view. Distinguishing between different viewpoints is first attempted (Stage IIB) then made genuinely but incompletely (Stage IIIA) is mainly mastered in Stage IIIB.

Piaget and Inhelder also put forward arguments that the ability to decentre, to see other viewpoints than their own, develops late. Donaldson (1978) claim otherwise. This point is also taken further later in the study.

Views supporting Piaget's research findings. Thirion (1969) looks at the drawing of geometric forms by children to see if topological or euclidean properties are preserved. He concludes that the sequence of development established by Piaget and Inhelder is confirmed but takes place earlier.

An intercultural investigation carried out by Jahoda, Deregowski and Sinha (1974) highlights the lack of research in the area of spatial considerations. They considered whether spatial-perceptual difficulties experienced in some cultures could be partly accounted for in terms of a persistence of predominately topological functioning. They found that their initial expectations had greatly underestimated the complexities of the issues and that there appears to be a considerable need for further research in associated areas.

Cohen (1978) examines the Scaling of Six Topological Piagetian groupings and comes to the following perhaps unsupported conclusions: Low rates of success on tests

> "might be overcome by providing greater opportunities for the children to have concrete experiences with objects as opposed to vicarious modes of learning. Many children spend little time with objects due to such other things as television. Many educators and psychologists have voiced their concern with respect to the possible negative effects on children of such endeavours. It is proposed that since children have less opportunity to manipulate objects, they have less chance of developing spatial concepts. Consequently, a child's formal educational experience should make efforts to provide occasions to compensate for these lost opportunities. Children should be encouraged to observe and draw many and varied types of things - should be encouraged to manipulate all sorts of objects, both of a familiar nature and also of a strictly geometric nature - not only to feel them, but also to observe them from a variety of perspectives and try to imagine what the object or group of objects would look like from many viewpoints".

Whilst some of these statements are a little oversimplified I believe the general remarks need emphasising. Laurendeau and Pinard (1970), Peel (1959), Dodwell (1963) and Lovell (1961) also, in a variety of ways support the Piagetian viewpoint.

Views questioning Piaget's research findings. Perspective ability has also been considered by Cox (1977). He looks at the abilities necessary to imagine or represent how objects would appear relative to each other from another person's point of view. He concludes that feeding back information to a child about another's point of view whilst he remains in his own place is more effective than the physical occupation of the other's place. This is contrary to Piaget and Inhelder's contention, though there appear to be occasions on which visual feedback is not less effective than verbal. Again, in the opinion of the investigator, there appears to be the need for further investigation.

A test of Piaget's hypothesis that a four year old child's perception of representational space is predominantly topological in nature was made by Martin (1976 a, b). He emphasises especially that the topological properties of a figure form a subset of the euclidean properties and that it is meaningless to talk of a given figure as being a topological figure or a euclidean figure. It is not a figure which is topological or euclidean but a transformation from one figure to another. There is considerable confusion on this point, its implications are important. Martin points out that mere replication of Piagetian tasks such as Lovell (1961), Dodwell (1963), Laurendeau and Pinard (1970) and Peel (1959) lead to results similar to Piaget's. Martin's results, however, do not support the theory that topological concepts develop prior to projective and euclidean in the child's representational space. He points out that in his experiment not one four year old made a drawing that preserved only topological properties. He claims that a theory such as Piaget's, because of possible implications (for the elementary mathematical curriculum) should be considered more thoroughly from mathematical as well as psychological viewpoints. Martin also delves deeper into Piaget's classification of shapes. He points out that Piaget uses phrases like "topological figure" and "euclidean figure" with the implication that figures are seen as being topological if they possess topological properties.

Martin shows that the classification into disjoint topological and euclidean figures is untenable. Hence, though it appears likely that Piaget's tests are very useful guides to children's spatial thinking, nevertheless his conclusions need to be interpreted with care. It also appears that some topological concepts develop before some euclidean but others develop afterwards. The whole place of projective transformational ideas in relation to other geometrical transformations appears to be very confused.

Geeslin and Shar (1979) consider a similar problem using a rather unusual approach. They used a battery of questions each of which contained a diagram above a line and two alternates underneath. Each child was asked to view the top figure and then to pick one of the two variants underneath "that was most like" the top figure.

Each of the ten items had one figure underneath which was not topologically equivalent but was "geometrically related" to the top one whilst the other is topologically equivalent but not congruent. A device, using a lattice grid, was evolved to give a measure for the amount of distortion necessary to change the top figure into either of the variants. The hypothesis was made that the figure with the least distortion would be the one chosen. With one exception the test items chosen appeared to support the hypothesis. It can then be argued that children do not look for topological equivalence but for some idea of how little the figure is modified.

Now there are problems with this approach. The measure of distortion depends on the idea of distance which is a notion in congruence. Hence, the measurement of distortion is not a topological measure. Nevertheless, the experiment takes a big step forward - a similar sort of process is used later in this study to measure distortions of a projective sort. It would be possible to compare figures which are projectively equivalent with those which are geometrically related or geometrically equivalent in a variety of ways.

Donaldson (1978) does not explicitly mention geometrical or spatial considerations as such. Nevertheless, spatial considerations are inherent in some of the experiments and could serve as a guide for similar work in perspective. The chapter headed "The ability to decentre" is of particular interest as it contains an account of a relook at Piaget's three mountains experiment. The conclusion is drawn that children have an ability to decentre - to appreciate someone else's point of view - greater than that maintained by Piaget, supporting Cox (1977) mentioned earlier.

Perception. It seems useful to consider projective geometry in relation to perception. The Piagetian view of perception is that it begins with the knowledge of objects resulting from experiential contact. It becomes representational when objects are known in their absence. Several aspects of perception seem worthy in this study of further consideration.

Perhaps the problems in perception are highlighted by seeing the difficulties some children have with the perception of pictorial depth. There is a large body of experimental evidence which suggests that some children who are not used to seeing pictures ("pictorially relatively unsophisticated subjects") who find it difficult to interpret cues which indicate depth in pictures and line drawings. (Deregowski, 1974). Deregowski suggests that giving experience of stereoscopic pictures may ease some of these difficulties. As children who are more pictorially sophisticated also have, in the main, to pick up these ideas of depth without specific teaching it is not perhaps surprising that depth cues are sometimes misinterpreted.

Perkins (1972) presents line drawings of parallelopipeds, (unfortunately using an affine transformation rather than a projective one, so that parallel edges are drawn parallel in the drawing rather than converging on a vanishing point). Subjects were asked to judge whether the drawings represented three dimensional rectangular boxes (cuboids). Now as angle is not an invariant in affine (or projective) geometry, none of the angles need be right angles and their probability of being so is presumably very small. On the basis of projective geometry, half the set of drawings could (but may not be) projections of a rectangular solid and the rest could not. Subjects tended strongly to judge as rectangular boxes only the drawings which could have been rectangular boxes. Thus the subjects perception seemed to impose perpendicularity on their angles.

The same conclusion in two dimensions was found by this investigator when presenting children with a perspective view of a chess board, (chapter 3). They did not see anything unusual in interpreting a general quadrilateral as a square or in being asked to find the centre of the square - whereas the centre of a general quadrilateral is a more ambiguous idea. Perkins states that a rectangular organisation is accomplished as far as possible short of conflicting with the rules of projective geometry. He suggests that the impetus towards reading right angles into pictures functions in co-ordination with other organising principles, such as depth cues.

Perkins also makes a plea for the exorcism of the word "illusion" from the vocabulary of perception as it conveys the unfortunate connotation of magic and deception. He suggests the use of the words "pattern anomalies" which do not, I feel, quite present the same ideas - perhaps visual paradoxes, visual surprises, or pattern confusions would be a better description.

Problems about visual illusions are not properly the concern of this study except in the way they impinge on projective ideas. Robinson (1972) contains a comprehensive collection of classifications of visual illusions and the research literature in this area is vast. Theories of geometrical optical illusions are not fully understood and some are contradictory, hence any conclusions drawn are rather tentative. However, although all photographs, aerial maps and many line drawings will be in perspective and an essential skill required of us all is the ability to interpret them. They are two dimensional images of three dimensional objects or views and, as such, are liable to misinterpretation.

Of interest, if marginally, is the study of Smith and Schroeder (1979). They suggest that nine year olds who had received instruction in spatial visualisation, outperformed those who had received no instruction. They also showed that among the slightly older, early adolescent children, only the boys significantly improved as a result of instruction. This suggests a point of critical learning and that there are important consequences in the timing of instruction in spatial ability. They also point out that it may be that spatial ability is important in mathematics and science and that the problem is complicated in that spatial ability is a cluster of skills.

A similar sort of conclusion could also be drawn from the study of geometric embedded figures. Ayers, Cannella and Search (1978) suggest that children in the early years of schooling were able to identify with relative ease embedded shapes, squares, rectangles and overlapping circles whereas others have found that children find embedded triangles more difficult. They conclude that it might be better to introduce early into the curriculum some other geometric shapes such as rectangles, circles and squares, and delay the introduction of the study of triangles. Although the study of triangles is an integral part of any geometric curriculum, their study tends to indicate that it might be a more difficult shape for children to understand. They make the sensible point that consideration should be given to the development of a geometrical curriculum co-ordinated for children of different ages as is instruction in number.

It would appear from these findings that children are able to cope with some euclidean ideas in their early years. This would, however, be an oversimplification. It is the investigator's view that the shapes chosen were somewhat arbitrary. There is an obvious need for further research into, for example, the perception of embedded parallelograms and quadrilaterals as opposed to squares and rectangles and embedded equilateral and isosceles and right angled triangles as well as more general ones.

Problems in perception may also be considered by looking at different interpretations of the word "size". The word "size" when used in descriptions of line drawings is very vague. These allow for different answers depending on the interpretations which may be made. Thus, the size of an object may be the actual size it has rather than the size in the drawing. This is usually referred to as the "physical size". Of lesser importance is the size as it occurs on the retina - the "retinal size" though it is unlikely that this would be referred to specifically in children's projective views. There is also a distinction between the "perceived size" and the "judged size" - the judged size being that which is given in response to a question, such a response might be "this seems bigger". The perceived size is the unobservable size which is related to the world of experience of judging apparent sizes against a background. A distant figure is seen small but might be perceived as a six foot man.

These meanings of the word "size" - which could be lengths, areas, and volumes presumably - highlight the problems of using the language of perception and the problems of interpreting of children's verbal responses. It is only too easy to draw incorrect conclusions. In a perspective view of a football field, for example, the question "Are the goal posts the same size?" is ambiguous. This investigator found (see Chapter 3) that some children, when asked, replied that the goal posts were not the same size because one was farther away than the other, that is, the drawing sizes were different (because presumably the physical sizes were equal). Other children replied that the goal posts were the same size because one was farther away than the other, that is, the physical sizes were equal (and, hence, the drawing sizes were different). Both answers to the question, yes and no, can be correct or incorrect depending on the way the wording is interpreted both by the questioner and by the one responding.

This brief review of research literature in projective geometrical learning highlights the complexity of the vast array of aspects to be considered - some of which conflict with others. The standpoints of the researchers are very different and they sometimes come to differing conclusions. The Piagetian research has led to a deeper understanding of the nature of some of the problems the child has in acquiring spatial-perceptual concepts. A great deal needs to be done before an overall co-ordinated projective instruction is available for the teacher in the classroom.

Projective Geometry : Its Relationships with other selected Curriculum Areas in the Primary School

Science. It has been observed previously that there is little tradition of science in most primary schools. In the last few years there has, however, been a great emphasis placed on the need for scientific learning for young children. The Schools Council Science 5 to 13 Project (1973) provides collections of different activities appropriate for particular age ranges. The publication of such material does not, of course, ensure its immediate adoption in schools. The project material has been translated into appropriate series of lessons. For example, Ambrose and Baker (1980); Ambrose, Baker and Salisbury (1981); Ambrose, Baker, Denny and Salisbury (1981) contain series of lesson plans which were produced, tried in classes with teachers present, and intended to enable teachers with a poor background in science to use them effectively.

Much of the work in science can involve interpretations of line drawings. Science 5 to 13 is full of such drawings, as are some of the Nuffield texts in science. Many of the diagrams are affine, the usual 30° isometric view, but perspective views are also used. Sometimes a drawing or a plan is given and sometimes a front or side elevation. In one text an isometric view of a test board in physics is given with realistic batteries, lamps, and connections drawn. One of the exercises is for children to draw a conventional plan type circuit using iconic and symbolic representations of the parts of the circuit. The transformation from one diagram to the other involves considerable changes though the topological properties are usually preserved. Such transformations require a greater degree of care which they do not always receive. Going up or down a dimension is also involved in interpreting drawings of chemical apparatus. Diagrams are drawn in perspective on occasions and in others as a conventionalised front elevation. Some exercises expect the construction of a three dimensional structure from written instructions and a front elevation drawn to scale. Perhaps a perspective line drawing would have given additional information for the child, and the teacher, of what the final structure should look like. Even this, however, is not always effective in providing sufficient information. Satterley (1964) makes this point in relation to maps. This is discussed further in the next section on geography.

Children's interpretations of diagrams given to them is not the only problem. A more acute problem occurs when children are asked to make their own drawings in science. Children may, reasonably, be expected to write up their experiments, either individually or as a group activity. They may be expected to report what they have seen, what conclusions they have drawn and the process could profitably involve drawings which could be more informative than a purely verbal account. It is reasonable to ask whether the drawings should be plans or front and side elevations, that is, two dimensional drawings, or whether they should be essentially an attempt to represent the three dimensional state of affairs by using either an affine (e.g. isometric or axonometric) or a projective drawing? Whatever is expected of them it seems only fair to give children some help. They are more likely to be satisfied with their drawings and use them more effectively to communicate information if they feel their diagrams are worth while. It seems that careful reflection on the sort of diagrams they might draw should be useful. (See a later section on Art).

It is remarked by Ambrose (1979) that children can often have difficulties with problems in science which are essentially three dimensional in nature. He suggests that those pupils who are reluctant to draw a sketch are the ones who make errors and find physics difficult. The use of plans, front elevations and more importantly, perspective diagrams at an earlier age might help. For example the problems of drawing front elevations of chemical apparatus such as test tubes, bunsen burners, flasks and so on could be considered when a perspective view is given originally. Part of the recording process in some biological or nature study could involve drawings or paintings of the children's collections, for example, of twigs and leaves. In such cases a deliberate use of a paper picture frame might be employed.

The interpretation of perspective drawings and of isometric drawings appears to be a very useful, if not essential skill in primary science activities. Equally the interpretation and construction of plans and elevations may illuminate ideas in physical science in particular.

Later in this study a projectively based curriculum is outlined. After considerable thought it was decided not to deal explicitly with scientific notions because the proposed materials might be rejected as irrelevant in a school where science is given a low priority. It is probably true to say that the need for interpreting drawing and constructing perspective, isometric and euclidean views is more likely to arise as the child reaches secondary school. Because my concern is with a projectively based curriculum for primary children, the selected activities are designed to be general enough to help with scientific activities but not to require extra curriculum time for science as such.

Geography. Perspective ideas and the use of maps have always been the concern of geographical educationalists. Most of these people have tended to work outside an Erlangen approach as the idea of the plurality of geometries is not generally available outside mathematics literature. Nevertheless, those who are concerned with the child's conception of space and of maps are now looking more towards Piagetian ideas than hitherto.

The use of aerial views was suggested as early as 1950, and it is recognised that map work causes children problems. Sandford (1973) suggests that very complex perceptual problems are presented to a child by a map. The superposition of lettering, embedded figures and various symbols may cause perceptual overload. This strains the immediate memory and hinders visual searching for information required from the map.

The phrase "cognitive map" is used by Catling (1978) and other geographical educators to mean an image in the mind of an area or region. Other psychologists use the term in a wider sense. Catling suggests that such cognitive or mental maps of areas, be they the layout of, for example, a house, a school, street, district or village, are images of areas which it is impossible to perceive from a single earthbound vantage point. Freehand drawings of an area might show in part the mental image which a child may have of an area. Such drawings change as the child's mental maps develop. He believes that there is a sequence of four stages of cognitive map representation: (a) Topological, (egocentric and unco-ordinated); (b) Projective 1, (quasi-egocentric, routes in plan form, buildings iconic, little perspective); (c) Projective 2, (quasi-abstract, routes continuous, some buildings symbolic, better perspective); and (d) Euclidean, (true map, routes and scale roughly accurate, few icons, highly symbolic). In a paper presented to the Geographical Association Annual Meeting in 1979 he suggests that teachers should foster map ability and that map reading, map drawing, the relationship between them, and the importance of psychological factors should be researched.

Gibson (1950) believes that three dimensional vision is primary and that two dimensional vision is acquired only with training and by adopting a special attitude. This has been taken further by Barufaldi and Dietz (1975) considering the effects of solid objects and two dimensional representation (both photographs and drawings) of the objects on visual observation. They obtained some significant results which were, however, rather inconclusive. They suggest, as a result, that children should be given greater opportunities to utilize photographs and drawings in the development of scientific skills because two-dimensional representations are basic to educational materials such as books, films, and displays.

Children's reactions to maps and aerial (direct) photographs were studied by Dale (1971). An investigation of the accuracy with which children recognised images on a map as against a direct aerial photograph showed that more features were correctly identified on the latter. Children's responses were strongly influenced by their previous experiences.

> "It was also significant that those children who appeared to have the clearest mental maps and went beyond the questions asked to point out other features in the village, were those who spent much of their time roaming around" (p. 176).

Having perceptual constancy, an ability to recognise an object out of its original context or from a different viewpoint, this group were able to read the map from any angle.

Satterley (1964) points out the great difficulties inherent in correlating oblique photographs and maps. Using maps in the field requires the ability to recognise land forms, to profit from spatial arrangement clues, and to co-ordinate two different viewpoints. He casts doubt on the value of map work in primary school. Willman (1966), however, makes the point that seeing is an active rather than a passive experience.

It appears that to many geographers a map means an Ordnance Survey (O. S.) map, often full of unnecessary clutter for the immediate geographical purpose. Contours, footpaths, parish boundaries, or such details are not always essential in finding the best road from one place to another. Generally it would seem that geographical educationalists have too limited a view of the child's cognitive map development. Even the Department of Education and Science (1978) has few topological or projective goals although item N specifies "visualising a landscape using photographs alongside a map".

Children draw different styles of maps depending on what they are required to do or what they wish to convey. Such a map may be in one style, topological, projective, affine or euclidean; or it may more likely be a mixture of these styles.

Art. The connections between art and geometry are problematical. The two areas are viewed so differently and the aims associated with work in the areas are so dissimilar that even obviously mutual relations may be ignored. Art is, in the main, designed to evoke the feelings and sensibilities of the beholder which has few guidelines of accepted rules or procedures. Although geometry may on occasions appeal aesthetically, unlike art it generates order and may be subject to proof. Few people have expertise in both areas and those considering inter-relations between art and geometry are likely to be mainly concerned with one subject and are, perhaps advisedly, rather tentative in their conclusions as affecting the other.

Nevertheless the representational aspects of art should not be ignored. As part of their education children need to learn to express themselves freely through a variety of media and on occasions they need to be able to attempt representations of things in the world as they see them. To a certain extent these representations may be of exclusive concern to the child, but they may also contain a communicative aspect. Children deprived of projective experiences may believe they cannot draw because in their own eyes their attempts appear to fail to satisfy others. Children develop a sensitivity towards other children's and adult's criticisms. The use of line drawings, paintings, and sketches to communicate ideas may be considered as a subset of the whole of artistic experiences and is called transactional art in this study.

Transactional art requires skills inappropriate in more obviously free expression activities. These skills may involve the drawing of objects, that is, representations as they are seen by the child in such a way that the drawings have the appearance of reality for drawer and viewer. To achieve a level of success an understanding of projections could be more than useful.

An interesting approach is taken by Ivins (1946) who shows that Greek thinking in art and in geometry are shown in their writings, sculptures and buildings. He defines perspective as the central projection of a three dimensional space upon a plane; as the way of making a picture on a flat surface so that it looks right from a single determined point of view. (This is really projection rather than perspective). Ivins concludes that the Greeks had little idea of perspective and that Alberti and Desargues were instigators in freeing Western Europe from a Greek tradition, providing a new logical and philosophical framework for thinking in art.

One of the interesting aspects of children's drawings is the information they give about their thinking. Goodnow (1977) points out that the graphic work that children create provides an insight into both their thinking and ours. She makes a strong plea for an integration of experiences; suggesting that drawing and thinking, the eye and the mind, the arts and the sciences, the soft and hard sciences should be regarded as interrelated rather than separate areas of experience. Graphic work is thinking made visible and as such has an important academic place at the side of numeracy and literacy. It can display thrift (that is, economy of effort and resources) and principles of organisation which are essential features of problem solving, and as such, graphic work should be part and parcel of the whole experience of children's lives.

Graphic work may therefore be used to provide information for the teacher about children's views and perceptions. Children who have some experiences of projections and perspective may have a more satisfying control over their drawings especially when these are intended to be representational or transactional.

Curriculum Development in Projective Geometry : Theoretical Considerations, Practical Problems, and Suggested Approaches

<u>Theoretical implications of the relationships between children's topology and projective geometry</u>. It is hardly possible to consider projective geometry without looking at some topological aspects. Later it will be argued that geometrical learning proceeds interdependently on a variety of fronts and that such learning need not be completely sequential and cumulative. Some topological ideas, however, may usefully have been met in some intuitive way prior to projective experiences. The following list is probably too exhaustive but has, I believe, a certain logic:

(a) The three dimensional nature of solids

(b) The two dimensional nature of surfaces

(c) A surface as a boundary separating two solids

(d) A surface as a partition of a solid into parts

(e) The one dimensional nature of curves (segments)

(f) A curve as a boundary separating two surfaces (regions)

(g) A curve as a partition of a surface into parts

(h) A curve as the intersection of two surfaces

(i) The zero dimensional nature of points (nodes)

(j) A point as a boundary separating two curves

(k) A point as a partition of a curve into parts

(l) A point as the intersection of two curves

(m) Positional aspects of points

(n) Continuity in solids, surfaces and curves

(o) Order of points on a continuous curve

It is probable that many of the ideas contained in this list would be inaccessible to some young children. The list is not intended as a teaching scheme, but as a focus on topological aspects which may arise from geometrical explorations of the environment.

These aspects will require a vocabulary e.g., solid, surface, region, boundary, separate, partition, part, space, figure, inside, outside, in, out, curve, segment, on, off, closed, open, loop, point, node, intersection, position, place, continuous, break, join, next, between, connect, after, and before.

A topological norm is a distance function; a qualitative measure of distances between points. When a norm is added to a topology the idea of size is incorporated into it. The vocabulary then extends to include: big (bigger, biggest), small, long, short, tall, holds more (less), covers more (less), high, low, near, far, and perhaps up, down, left and right. Piaget suggests that speed comparisons are easier to comprehend than distance implying extra vocabulary such as fast (faster, fastest) and slow, early (earlier) and late. The addition of the concept of a norm takes topology a step nearer projective geometry.

It is not suggested, however, that the child's learning develops through topology without, and then with, a norm to projective and euclidean ideas. Young children can recognise that their fathers are taller than they are without going through various topological stages of development! Without the idea of a distance, it is difficult to lead on to projective experiences. Again distance is a euclidean concept rather than projective but without such it is difficult to explore the children's projective thinking. To give an example of this, eight year old children were asked to consider an oblique projective line drawing of a football field. They were asked which of the two goalposts - one near and one further away - is larger. Now this question is rather naively ambiguous. Do we mean the real goal posts the drawing represents? (the physical size). Do we mean the line representing the goal posts on the piece of paper? (drawing size). Do we mean the size which the child considers relating the size to his or her world of experience? (perceived size). Nevertheless, the question enabled the children to focus on a relevant problem - one which could not be asked in purely projective terms. The follow up question asked them to write a short account of why they thought a goal post was larger than another and here it was possible to see something of their interpretations of size and their topological interpretations.

The distinctive nature of projections is that of mapping a three or two dimensional picture onto a plane. For example, a child's drawing of his journey from home to school may be considered as a model of the real world journey that he takes. It is a two dimensional model of a three dimensional situation. Mapping down a dimension from three to two dimensions has problems. Technically such a mapping is many-one, that is, many different points in space become the same point on the plane. Such a mapping does not have an inverse (the reverse mapping is one-many) and this is the reason why some visual illusions exist. This may be explained using the visual illusion known as the Necker cube.

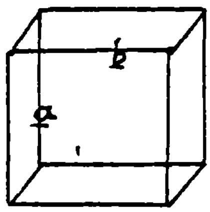


Figure 7. A Necker cube, with a and b as skew lines.

Here the representation of the cube preserves the parallelism of the figure. The interpretation of the figure differs depending on whether line a is taken as being in front of line b or behind it. The two skew lines a and b, when represented on a plane (mapped down a dimension) have a point of intersection. Two points (in fact a line of points) on the cube become the same point on the line drawing. The problem is one of mapping up a dimension: of interpreting a line drawing as a three dimensional object, such a mapping is not unique. Further investigation of this illusion is contained in the third chapter.

Projective transformations require the image set to be a plane or a subset of a plane. From this a line is found as the intersection of two planes. This may be shown in various ways: (a) By paper folding (whereby two planes meet in a line and two planes are partitioned by the line), (b) By line of sight (this is a concept of straightness which does not appear to come from flatness but to be intrinsically one dimensional), (c) By having a flexible curve and pulling it tight (a straight line being the shortest distance between two points) - that is - relying on distance ideas with the line not being considered as a subset of a plane).

Here I have raised two red herrings which do not need to be considered in any great detail. One is that a taut string will actually form a very shallow catenary due to the weight of the string which is of little importance in this context. The other that in relativistic terms the shortest distance definition is only a first approximation. Neither of these considerations need bother us any further here.

To deal with projective geometry and its properties the invariance of cross ratio is required. This is considerably more difficult than the idea of a straight line. Cross ratio is usually defined in terms of an invariant property of congruent geometry, that of length. More than this, it entails showing the invariance of a certain ratio of ratios of lengths.

As straightness is a projective invariant so polygons remain polygons under projective transformation. Hence, projective invariants include triangle, quadrilateral, and n-gons in general. Projective activities for children should include the experience and construction of polygons, starting with simple cases and leading to more complicated ones.

The main invariant in projective geometry is usually considered to be that of straightness. This is, of course, a great distinguishing feature between topological and projective transformations. This is, however, not the whole story as the concept of a plane is absolutely fundamental. The idea of a flat surface seems usually to be taken for granted and it may well be that implicit teaching is sufficient. As far as I have been able to ascertain no one has investigated flatness in the way that Piaget et al have investigated straightness. This is probably due to the difficulty of setting up reasonable apparatus for discussions about the whole idea of a plane surface.

There may be more of a problem here than might be supposed. One test of flatness would be to have two surfaces, put them together, if possible, and then arbitrarily move them in relation to each other. Only if they remain in contact with each other everywhere will the surfaces be flat. This test appears to have the difficulties of some of Euclid's postulates and perhaps a few more. There are certainly problems about considering an infinite number of points or an infinite number of rotations and translations. But despite the theoretical difficulties there is a readily understandable practical reference to glass papering to produce a smooth flat surface. It might be called the "glass papering" test for flatness. Another test, perhaps the more usual one, is using one surface and one line and testing that the line lies in the surface in all positions. The same sort of problems arise, however, as in the previous test. This may be called the "carpenters test" (rule?) of flatness. It seems to me that if it is proper to proceed down the dimensions introducing solids before surfaces and these before curves and then points, that it is better to introduce flatness prior to straightness. (The three dimensional analogue presumably does not exist).

It should be mentioned that there are difficulties in using such a common word as "flat". It is like "big" which can mean different things in different circumstances. A joke can "fall flat" and a statement may allow no contradiction as in "I tell you flat". Children may "feel flat" the day after a particulary exciting event took place or they may run "flat out" across the bumpy field. They may live in a flat or meet certain cuboids in the Dienes' Multibase Arithmetic Apparatus which they are told to call "flats". They may consider that a wall is not flat because it is vertical whereas flat means horizontal. A flat surface in normal language means one which is not sloping or tilting more often than one which is even or plane. The word "plane" also may be confused with "plain". In reply to a verbal question a child may consider a wall decorated with paper as pretty rather than plain. The ambiquity does not need stressing. Problems of this kind occur in other subject areas.

An interesting sideline may be mentioned here. If children did learn their geometry from topology to euclidean then similarity should preceed congruence. The implication is that angle should be introduced as a concept prior to length. Although geometrical learning may not be completely sequential it may be that an overemphasis on length measurement in the presecondary school makes the acquisition of an angle concept more difficult and that angle should be introduced before or even simultaneously with length, this is outside my brief however.

Some other theoretical considerations are suggested by Burn (1975) who develops a hierarchy of theorems about geometrical transformations emphasising relationships between geometrical axioms and number systems. Like Choquet (1969) his approach is essentially affine. He defines a projective plane and shows that every affine plane may be embedded in a projective one. He uses a finite, residue class approach which may be applicable in the presecondary school in a modified form. Land (1960) discusses Golden sections and other problems which are marginally projective.

Morley's theorem, though euclidean, is interesting and may be used to develop ruler and protractor skills. Escher (1967) has some interesting things to put forward about congruence and projections in his work. His distortions of perspective to make nonsense pictures, such as water in perpetual motion down hill, make interesting visual illusions.

Practical Problems

The role of the teacher of geometry in the primary school. In discussing secondary education, Tammadge (1981) states that teaching is demanding and calls for special qualities. In mathematics teaching this is particulary true as the subject is intrinsically impersonal. The mathematics teacher must contrive situations to establish interpersonal relations, to spur curiosity, and to provoke controversy. It is this investigator's opinion that the same holds in the primary school, perhaps more so when many teachers feel themselves to be inadequate mathematically.

It is only in those primary schools in which there is a mathematical presence, by which I mean that there is some member of staff who has a particular interest or expertise in mathematics, that an attempt is likely to be made to develop a comprehensive and organised set of mathematical activities. To be effective, this mathematical presence has to influence the teaching and learning of mathematics throughout the school. Unless professional relationships are good, non mathematical teachers develop strategies for ignoring mathematical innovations, or merely pay lip service to proposed schemes and introduce them without enthusiasm or understanding.

There is also the matter of parental demands to be considered. Parents expect their children to learn and be skilful in the performance of arithmetic algorithms. Teachers may well heed these pressures as they see the parental views of mathematics as fitting their own views. Teachers may use children's performances in arithmetic algorithms as their only measure of their skills as teachers of mathematics.

The teachers may also have a euclidean view of geometry and see it as irrelevant or, at best, as an unnecessary addition to the curriculum. Many of them will have met geometrical ideas in a pseudological framework as something irrelevant outside, or even inside, the classroom. The secondary school emphasis on the operations of formal structures which typifies geometry is seen, quite rightly, as inappropriate for primary children. Teachers may also have found geometry to be a distasteful, uninteresting, and difficult subject which they never felt they mastered. In primary schools teachers tend to teach from strengths and often geometry is not one of these.

It has been suggested by Skemp (1963) and others that children, (and adults), can be separated into mainly visual or verbal thinkers. Some children are able to understand better by visual encounters (I see and I remember) and through tactile experiences (I do and I understand); rather than through verbal instruction (I am talked to and I do not retain!) Primary school children are required to express their thinking in verbal terms in their reading and writing as this is an essential mode of communication. This emphasis on verbal communication might cause many primary teachers to have a bias towards verbal thinking and hence have a tendency to neglect visual aspects of children's learning. The primary teacher who is strong mathematically, has an Erlangen viewpoint and is a visual thinker may be rare. Somehow the situation needs to be changed. Teachers need to see primary geometry as essential, exciting, and as providing many links with desirable activities in and out of the classroom. They need to be encouraged to give their children opportunities to be visualisers as well as verbalisers.

To present geometry in a purely euclidean form is to ignore recent advances in geometrical thought. Mathematics did not stop at the end of the classical Greek era or even at the end of the nineteenth century.

Recent advances in geometry, however, tend to be conceptually difficult and unsuitable for children, as indeed are some euclidean ideas.

It seems more sensible to consider spatial concepts and to decide which of them are suitable for teaching in the primary stage. Teachers and children use spatial ideas in many areas of the curriculum and they are involved in many of the general activities of young children. We are three dimensional creatures living in a three dimensional world and are often curious to explore our world tactually and visually. We need to decide what is suitable for children at a particular stage of development, of a particular age, in a particular class, in a particular environment. It is important to decide what spatial concepts the individual has and should possess and to decide the most effective methods for learning them. Children, with or without the help of their teachers, are continually exposed to visual information which they are interpreting, partially interpreting or misinterpreting. It is the contention here that they need all the help they can get and that there is a need for teachers to have a resource package of ideas and activities which are projectively based. It is very difficult in geometry to estimate the level of experience, understanding and commitment that a child brings to an activity, so perhaps the greater the variety of activities, the greater will be the chance that some worthwhile learning may take place. Diagnostic exercises are an essential element of my curriculum materials and these should enable teachers to understand some of their children's conceptual problems more effectively. Any work asked of children should be such that they can begin the task with a reasonable prospect of avoiding the frustration of failure and of being satisfied with their achievements.

Suggested approaches. From a simplistic Erlangen view a programme would consist of: (a) Topological ideas (cognitive maps and drawings, the children perceiving the environment in topological terms); (b) Projective ideas (flatness, straightness, polygons, and ruler skills); (c) Affine ideas (isometric diagrams and visual illusions); (d) Euclidean ideas (a tessellation approach to similarity and congruence i.e., shape and size).

A variation of this is given in appendix 3 which is a discussion document which emphasises a spiral approach to the learning of geometrical concepts. It hints at a looser sequencing of geometrical learning. It is a recurring theme in the following chapters that geometrical learning should proceed interdependently on a variety of fronts.

CHAPTER THREE

Children's Projective Geometric Abilities

An Exploratory Phase of the Research

Straightness as a concept. As a starting point it was decided that, to obtain experience in investigations, children's concept of straightness was a suitable topic. Some of the pitfalls in investigating children's ideas are well documented. Barber (1979) makes the point that it is easy for an investigator to work within his own paradigm with the result that the questions, answers, and responses are contained within his preconceived ideas. Another danger is that imprecise instructions may intentionally or unintentionally lead to suspect conclusions.

With the above in mind, a tráal investigation was set up to explore problems in setting up and classifying data using a modification of the telephone experiment of Piaget and Inhelder (1956, p.156). The modification enabled teachers to conduct the experiment with a class or group rather than an individual. The experiment involved children in placing two cuisenaire rods upright on two prescribed outline squares on a sheet of paper and then placing three more rods on the page so that all of them were in the same straight line. The experiment was repeated using a different border as outlined on the paper: on one sheet there was a heavy rectangular border, on the other a heavy irregular curved loop. Teachers who were to implement the investigation met and discussion took place to examine possible pitfalls. Concern was expressed about the ambiguous nature of the brief written instructions as given in figure 8. Poor readers also might need special consideration. The results that were obtained were, unlike those of Piaget and Inhelder, quantified. A score, for each individual child's response, was obtained by finding the deviation from an exactly straight position for the rods. This was obtained by measuring the deviation in half centimetres (0.5 cm) of each

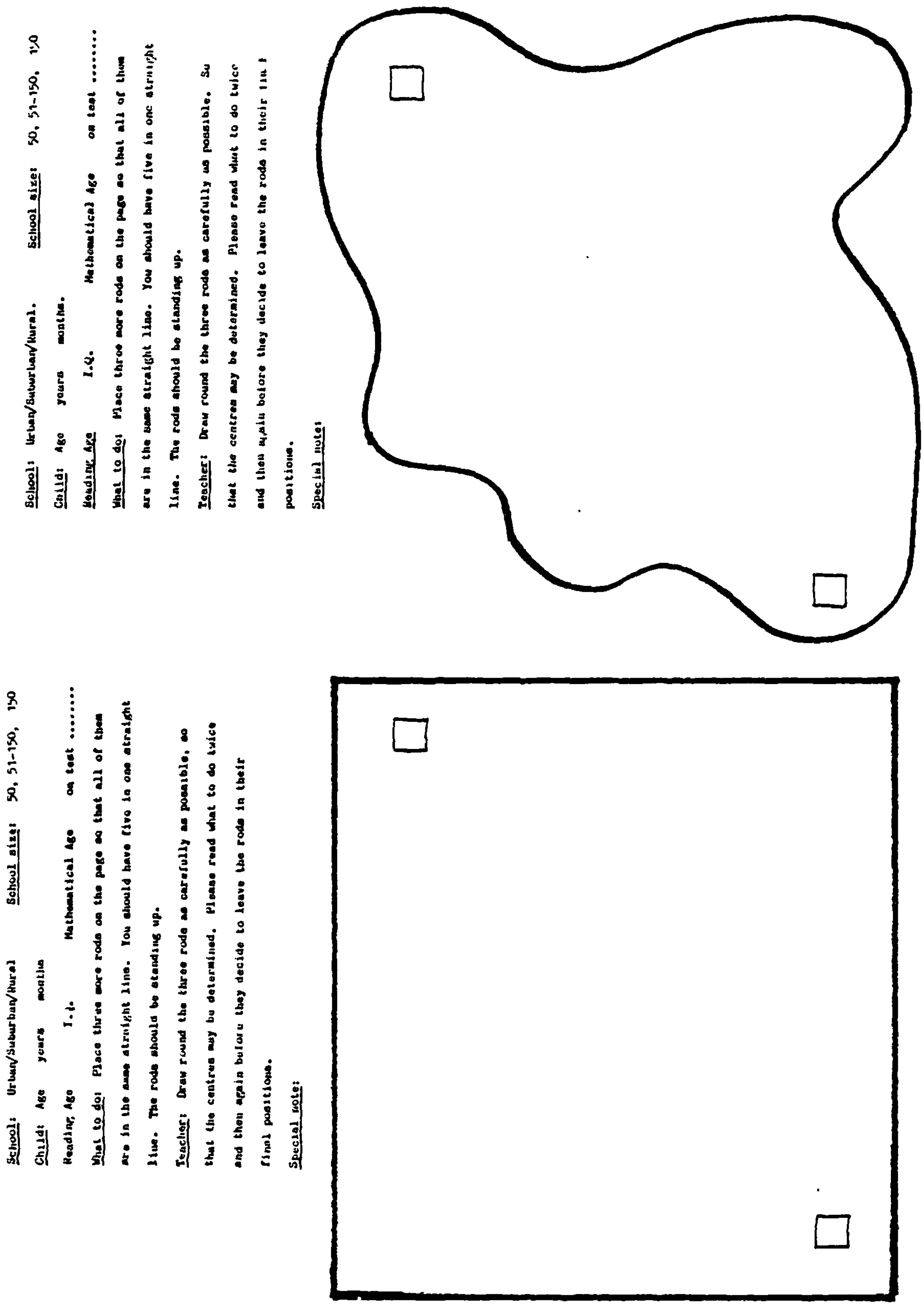


Figure 8 Straightness experiment using (a) a rectangular border and (b) a curved border

of the three rods and summing them. For example, if a rod were out of line by 3.1 cm this would be scored as seven (that is, six half-centimetres and a further one for the extra 0.1 cm). Thus, if all the three rods were perfectly lined up a score of zero would be recorded, but a slight deviation would result in a score of three, one for each rod. A score of three or less implies straightness.

Only twenty seven children were tested, a few from several classes, in age ranging from 4 years 3 months to 9 years 11 months.

Several problems were exposed. It was evident that some children misunderstood the instructions as they merely placed their rods in a straight line (thus exhibiting a knowledge of straightness) but ignored the line determined by the two original rods. The results were as given in Table 1.

Table 1

Straightness scores for (a) Rectangular Border and (b) Curved Border Experiments

	Age	4		5		6		7		8		9		TOTAL	
	Total deviation in 0.5 cm batches	a	b	a	b	a	b	a	b	a	b	a	b	a	b
random placing	50–59	0	0	0	0	1	0	0	0	0	0	0	0	1	0
	40–49	0	0	2	0	0	0	1	0	0	0	1	0	4	0
	30–39	0	0	0	0	0	0	0	0	0	0	0	1	0	1
	20–29	0	0	0	1	0	0	0	1	0	0	0	0	0	2
	10–19	1	4	2	3	0	1	0	0	0	0	0	0	3	8
straight placing	0–9	4	1	6	6	2	2	3	3	2	2	2	2	19	16
	TOTAL	5		10		3		4		2		3		27	

As may be seen from the table, the results were somewhat unexpected. Bearing in mind that the sample of children was small, the younger children, contrary to normal expectations, performed better than the older. The effect of the two different borders appears to be minimal. The inferences which may be drawn from this experiment are slight. There is a possible suggestion that the concept of straightness is formed earlier than might be expected. The chief value in the exercise was the experience gained by the investigator in dealing with experimental evidence.

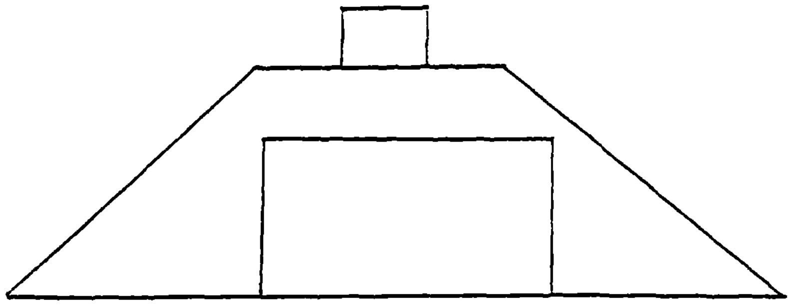
Interpretations of line drawings. As part of the exploratory phase in this research it was decided to try out some projective materials in school. The school selected was a city school with mixed ability parallel classes from a varied social background. This provided flexibility as a variety of options were available in selecting children for testing at varying stages of the research as might prove necessary. Available information about children's projective abilities made the exploration very tentative. The little information that was available seemed to suggest that eight year old children would be most likely to produce the most useful data, but the possibility of using older children was always kept in mind. One class of eight year olds was chosen. Although in her probationary year the class teacher was an excellent teacher, a previous student of mine, and aware of my general approaches to mathematical education. For most of the exploratory research we worked closely together within the research framework. A teacher of initiative and resource she continued the work in my absence and on many occasions made her own distinctive contributions.

The activities required children's responses in writing and by drawing on prepared work sheets. The instructions were put on the sheets and read to the class as well. The work sheets are in figures 11 to 17. The numbering was made deliberately intermittent to allow for additions to be made if this was necessary. To avoid the work being seen by the children as an additional mathematical burden each worksheet was given the neutral heading "some questions". It should be noted that A14 and A15 were not

Some Questions AJS A1.

Name ______________________

A/_ B/G A B C D



"Here is a view of a football field.

The view is from behind one goal" Repeat.

1. Mark one of the corners of the field with an X.
2. Draw on the diagram one corner post with its little flag.
3. Put a letter A on the near crossbar.
4. Put a letter B on the far crossbar.
5. Are the two goals the same size? Put yes or no in the box here.
6. Write why you think that is so ______________________

____

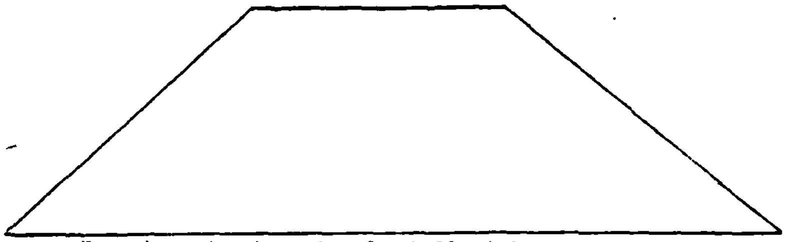
____

Figure 9 A football pitch (A1)

Some Questions AJS A2.

Name ______________________

A _/_ B/G A B C D



Here is a drawing of a football field.

It is being looked at from behind a goal.

1. Draw in the corner posts with their flags.
2. Put a cross X at the centre, the middle, of the field.
3. Draw in the centre circle.
4. Draw in the goal posts at the far end of the field.

Figure 10 A football pitch (A2)

<u>Some Questions</u>   AJS A5.

Name ______________________

A _ / _   B/G   A B C D

Here is a picture of some fields seen from a helicopter.

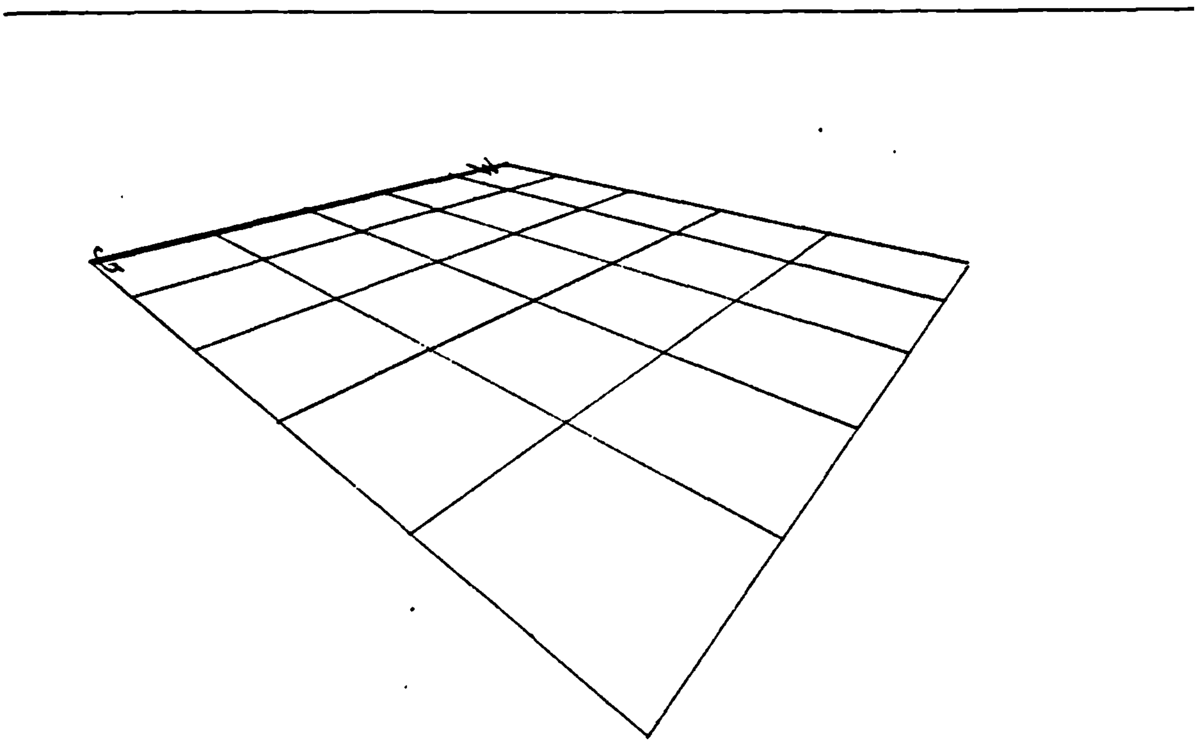


Figure 11   Some fields (A5)

Some Questions AJS A5 instructions.

Name ______________________________

A _ / _ B/G A B C D

Here is a picture of some fields seen from the air.

1. Put the letter A in the square nearest the helicopter.
2. Put a letter M in the middle square.
3. Are all the squares the same size? Answer yes or no in the box on the paper.
4. Say why you have said that.
   Write your answer along the line on your paper.
5. Draw in an extra row of squares along the edge marked with a thick line - this thick line has G.W. written on it.

Figure 11 Continued

Some Questions AJS A6.

Name ________________________

A _ / _ B/G A B C D

Here is a picture of some fields seen from a helicopter in the air.

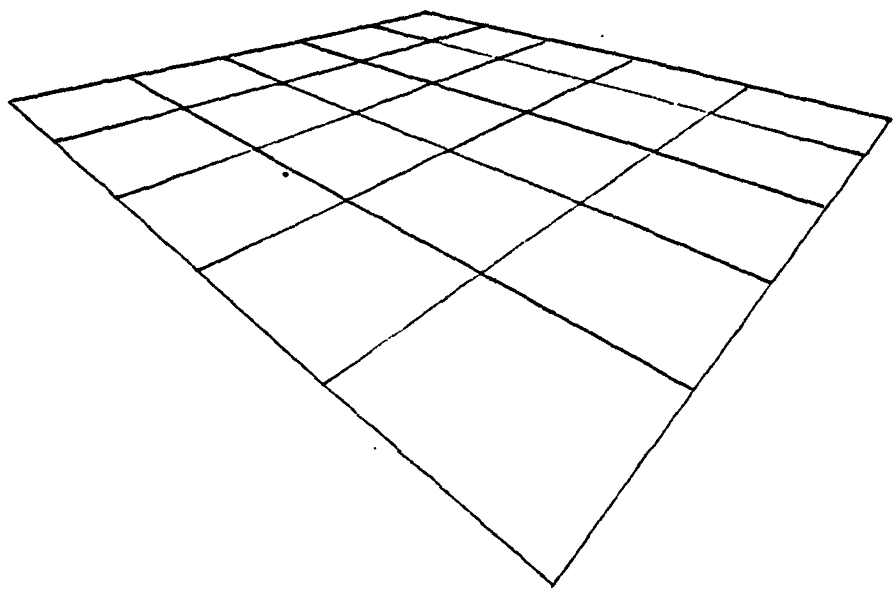


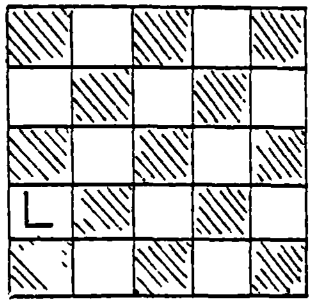
Figure 12 Some more fields (A6)

Some Questions   AJS A6 instructions.

Name ____________________

A _ / _   B/G   A B C D

1. Put the letter H on the horizon away in the distance.
2. Shade in the middle square.
3. Now shade in the squares round it just like a chessboard design like this one here.



4. Mark on the picture the square marked L.
5. Put an X at the middle of this square.

Figure 12   Continued

Some Questions AJS A10

Name ______________________

A _ / _ B/G A B C D

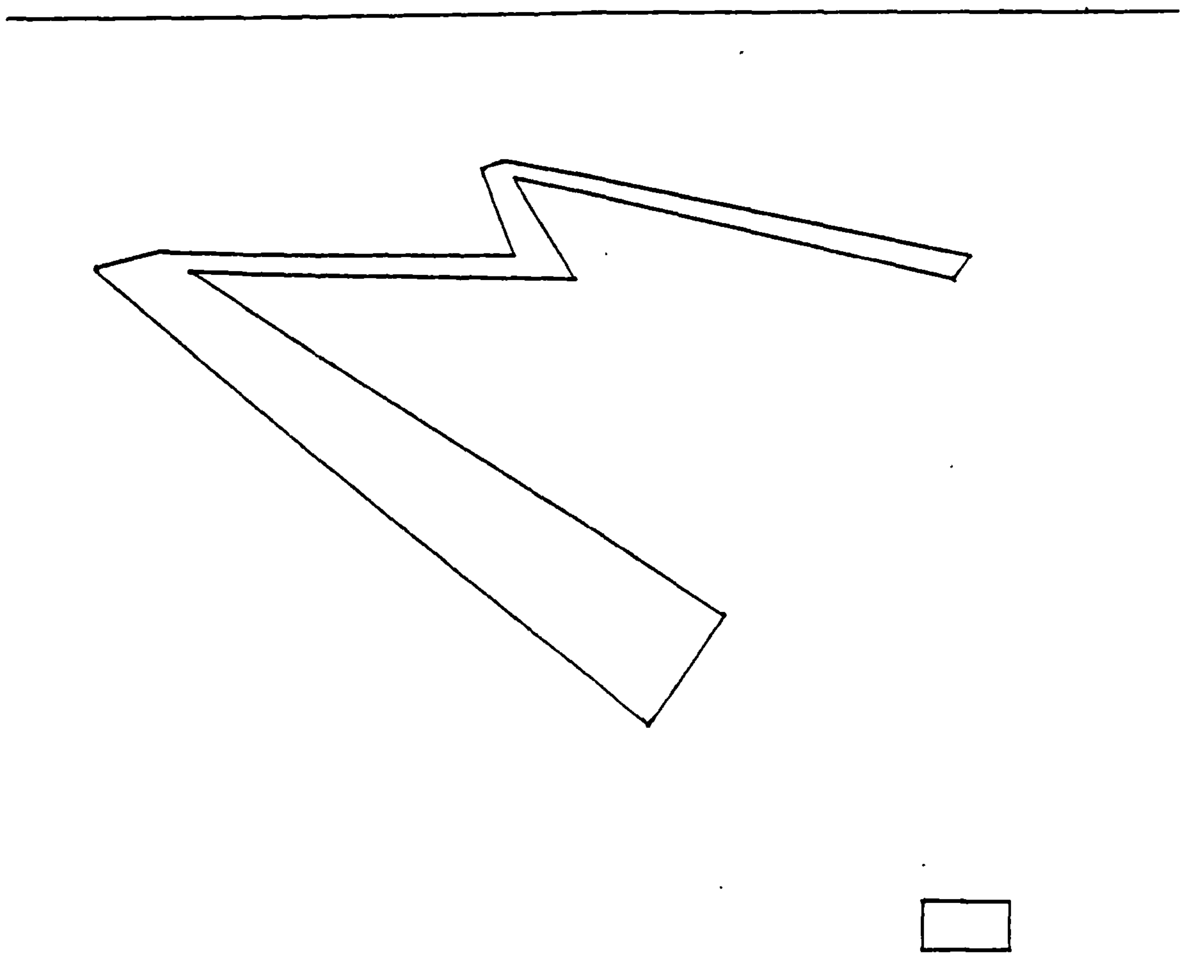


Figure 13 An airfield (A10)

Some Questions AJS A11.

Name ______________________

A _ / _ B/G A B C D

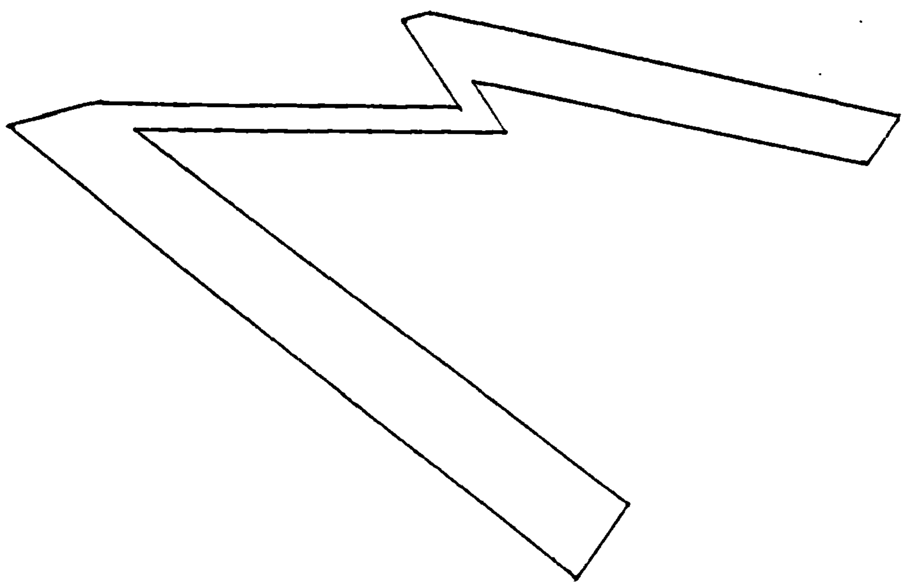


Figure 13 continued An airfie'.d (A11)

Some Questions AJS A10, 11 instructions.

Name ______________________________

A _ / _ B/G A B C D

A pilot is flying a helicopter over a large airfield and he sees a letter painted on the ground.

Look at these two pictures and write on them which letter is painted on the ground.

Now you have two pictures. One looks right and one does not look quite right.

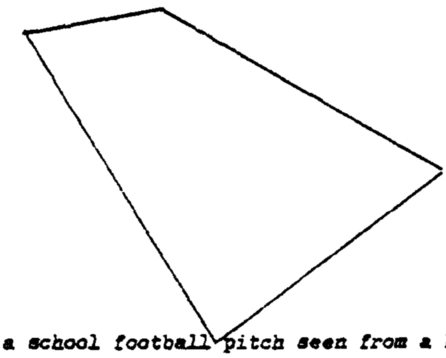
Put a tick in the box on the one which looks right, it looks like a photograph taken from a helicopter.

Figure 13 Continued

Some Questions AJS A14.

Name

A _ / _ B/G A B C D



Here is a school football pitch seen from a helicopter.

Away in the distance is the horizon.

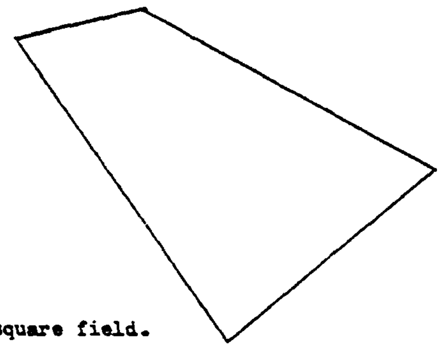
1. Put a letter G where the near goal post should be.
2. Put a letter M right in the middle of the field where the ball is put to start the match.

Figure 14 Another football pitch (A14)

Some Questions AJS A15.

Name ______________________

A _ / _ B/G A B C D



Here is a square field.

It is being looked at from a helicopter in the air.

1. Put the letter H on the horizon at the top of the picture.
2. Put the letter N on the nearest corner of the field.

   Here is the field seen from directly above.

3. It has been divided into four small equal squares.
4. Divide the square on the picture into four small squares and shade two of them as in the diagram below.

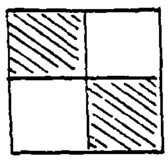
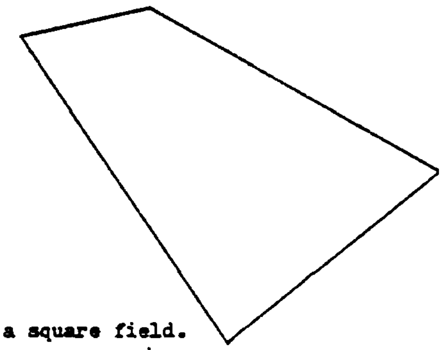


Figure 15 A square field (A15)

Some Questions   AJS A16.

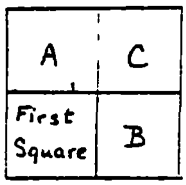
Name ____________________

A _ / _   B/G   A B C D



Here is a square field.

It is being looked at from a helicopter in the air.



This first square has three other squares placed round it. They are marked A, B and C.

Draw these three squares on the diagram at the top of the page.

Figure 15 continued   A square field (A16)

<u>Some Questions</u> AJS A20 instructions.

<u>Name ______________________________</u>

A _ / _ B/G A B C D

Here is an unusual game of noughts and crosses.

You can only use the ten big dots marked on the diagram.

The game is played just like noughts and crosses.

You take it in turns to put your noughts and your partner puts his crosses.

A player wins if he gets TWO rows of noughts or TWO rows of crosses.

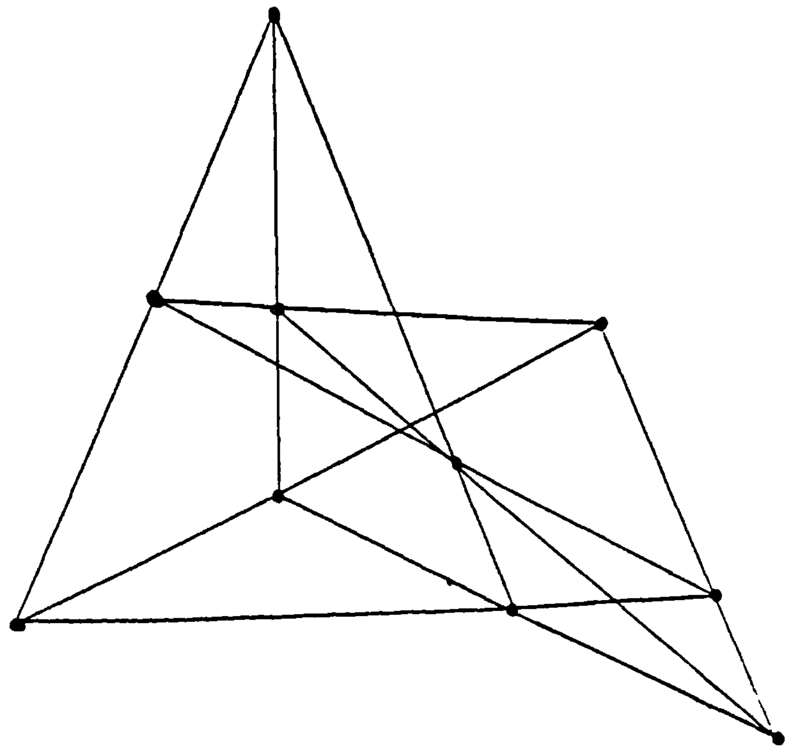


Figure 16 Desargues' noughts and crosses (A20)

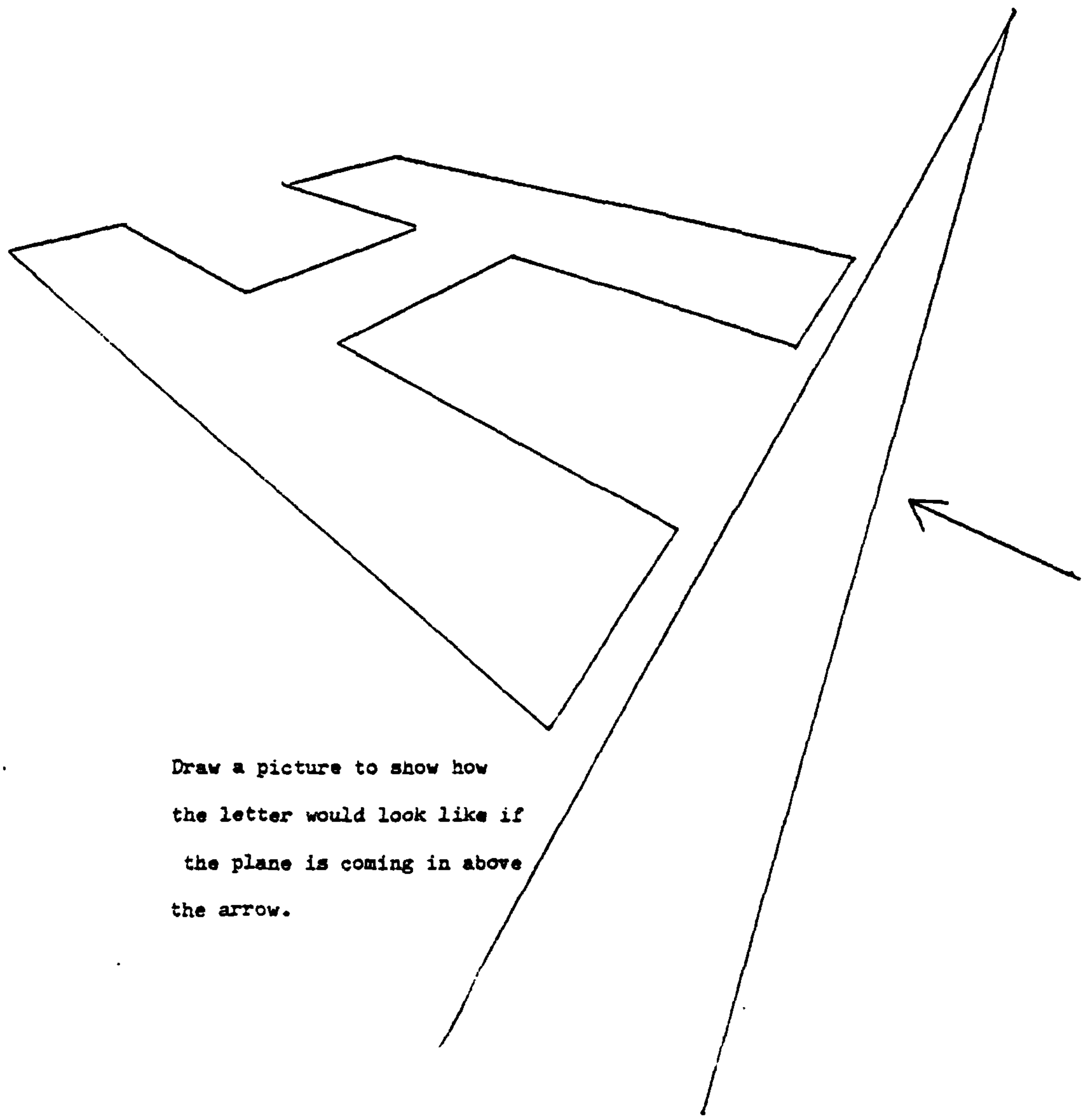
Some Questions AJS A23.

Name ______________________________

A _ / _ B/G A B C D

An airplane is coming in to land on the runway. By the side is a large letter. Write this letter in the box.

The letter is $\boxed{}$



Draw a picture to show how the letter would look like if the plane is coming in above the arrow.

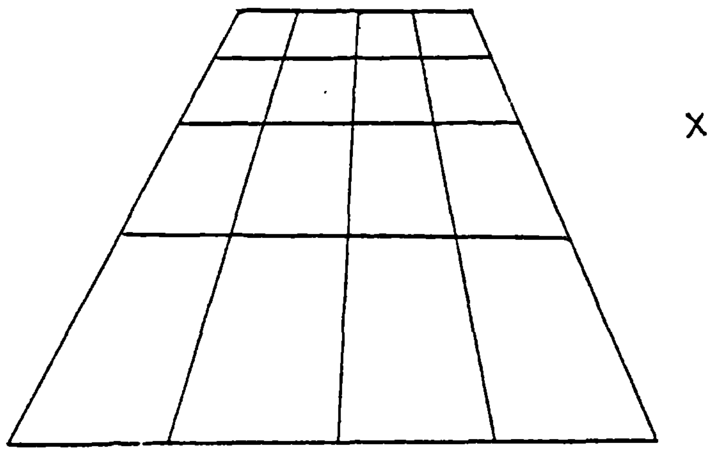
Figure 17 Another airfield (A23)

Some Questions AJS A26.

Name ______________________________

A _ / _ B/G A B C D

Here is a picture of a tiled floor.



Draw a picture to show what the floor would look like to someone standing on the place marked with a cross - use the sheet marked A27 for your diagram.

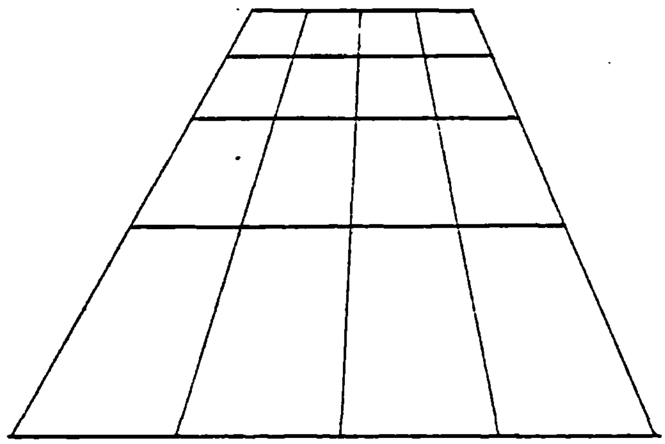
Figure 18 Tiled floors (A26)

Some Questions AJS A29

Name

A-/- B/G ABCD

Here is a picture of a tiled floor.



On sheet A30, draw a picture as though you were looking at the tiled floor from the place marked with a Y.

Y

Figure 18 continued Tiled floors (A29)

Some Questions AJS A30

Name ______________________________

A-/- B/G ABCD

The tiled floor seen from Y looks like this

Y

Figures 18 continued Tiled floors (A30)

used and that A20 was amended before use. Prior to the start of the investigation, the children were categorised by the teacher into four ability bands, on the general abilities displayed by children in other school work. These are labelled A for the top quarter, B, the next quarter, C, the next quarter, and D, the bottom quarter of the range in the class.

The results of the investigation are given in app 4 with ✓ implying a correct response or a response close enough to be considered correct. In some questions such a decision is automatic but in others it is only possible to be more subjective. On occasions the decision was so difficult that it was not made and a ? given instead.

A1. Question 1 (29), twenty nine correct responses from thirty one children, question 2 (28), question 3 (24) and question 4 (23) gave most children a good start. Question 5 (not scored) and question 6 (16) should be considered together. The problem of the use of the word size is discussed in some detail in chapter 2.

A2. Question 1 (29) was an easy start. Question 2 (4) proved too difficult for eight year olds most of whom put the centre too close to the near side of the field. Not one child managed to make a reasonable attempt at drawing the centre circle. Question 4 (16) gave interesting results including drawing the far side posts upside down 'to keep them on the pitch'? Over half, however, made an attempt which could be called correct.

A5. Questions 1 (30) and 2 (27) were well answered but Questions 3 (not scored) and 4 (14) caused the expected problems as exhibited in A1. Question 5 (8) resulted in the majority of cases in a "Piagetian" response, very reminiscent of the ink bottle test. The attempt was usually made to draw a square "face on" and they try to draw it flat on the paper in perspective. The result was in many cases to make the new squares appear to be on a hillside somewhere between the flat view required and the face on view which is the usual one for a square. Not one subject continued the edges of the squares straight on to help produce the next row of squares.

A6. Question 1 (7) shows that the idea of a horizon is not usually present at this age. By observing the children at the tasks, here the shading is partly topological or effected by an intuitive co-ordinate system, for example, children counted one square along and then one square up and shaded this square.

A10 and 11. This question was badly constructed. The diagrams were too similar to each other. The responses did not correlate closely with ability (see appendix 4), and it appears that the responses were quite random. Question 6. A16. Questions 1 (0) and 2 (0) proved too difficult an exercise.

A20. is a projectively based game. It appears to be a game depending mostly on logic. It was thought important to introduce an element of play into the exploratory research. A longer account of this exercise and some useful modifications is contained in Salisbury (1982).

A23 and A26 to A30 involve the ability to decentre in a very abstract situation and most children had, as would be expected, considerable difficulty. Children tried to draw lines perceived as parallel in reality as actually parallel on the paper. An interesting point is the extra difficulty imposed by the boundary in A28 (1) compared with A27 (14).

As the investigation progressed and the information from the children's responses was analysed, interesting as it was, it did not give me all the data I was seeking. This was chiefly due to the exploratory nature of the exercise. It was decided to proceed on two broad fronts. A tentative curriculum was being assembled as a result of the research findings and the responses to the exploratory investigation as a second phase in the research. An account of this is contained in Chapter 4. The other broad front was research into children's preferences rather than their interpretations of line drawings.

Research into Children's Projective Preferences

Outline of research background. The wide divergence of views on children's geometrical learning is considered in Chapter 2. It is difficult to reconcile the views of Piaget and Inhelder (1956), Martin (1976 a, b), and Geeslin and Shar (1979). The investigator's attention was focused on the 'topological' and 'euclidean' figures of Piaget and Inhelder (p. 54). Their emphasis appears to be placed on considering the figures as an assemblage of lines and curves rather than their inherent 'shape'. Geeslin and Shar also emphasise the boundaries from which their figures are derived rather than the interiors. Closed figures, in particular, may also be considered in terms of their interior, the 'shape' or 'area' that they have rather than the way the boundary is constructed.

The research design. It was decided to have several polygonal figures and to transform them in a variety of ways. Children would then be given a figure and asked to decide on a preference from alternative transformations of the original. A projective transformation was used and compared with six other transformations. Six different figures were used giving thirty six preference choices. The six figures were asymmetrical, reasonably dissimilar to each other and neutral; that is, they are not easily described by reference to another more familiar figure. The two least neutral were the "backward F" and the "flag". With the exception of the "flag" they were all composed of unit squares. These were included to give the exercise some interest for children, otherwise the whole exercise might have been seen as quite meaningless. (None of the children did, in the event, give up before the end or rush to get the exercise over as quickly as possible). The figures were chosen as shown in figure 20.

Each of the six figures is transformed in seven ways altogether. The first transformation is a projective one which leaves the orientation unchanged. The base of the figure remains horizontal and the top appears to recede. This is the basic transformation against which the other transformations were tested. It was called the standard projective equivalent. (S.P.E.) The six other transformations were: (a) An oblique projective equivalent (from a different viewpoint). (b) An affine equivalent (a shear to the left giving the verticals a gradient of $^{-}5$). (c) A similar (a linear magnification of 1.5). (d) A reflective equivalent (in a vertical axis). (e) A rotational equivalent (through a right angle anticlockwise). (f) A "piecewise congruent" (translating one part of the figure relative to the other). As in Geeslin and Shar's study a figure was presented and the child asked to say which of the alternatives was most like it. In each case the standard projective equivalent was one of the choices. In all, this entailed thirty six selections using six figures P through to U for each of the six transformations (a) through to (f). A Latin square arrangement was chosen, isomorphic to the isometries of an equilateral triangle, as in figure 19.

(a)	(b)	(c)	(d)	(e)	(f)
P	Q	R	S	T	U
Q	P	U	T	S	R
R	T	P	U	Q	S
S	U	T	P	R	Q
T	R	S	Q	U	P
U	S	Q	R	P	T

Figure 19. Randomisation of preference alternatives using a latin square arrangement.

In some items the standard projective equivalent (S.P.E.) was on the left and in others on the right. The choice of these was randomly determined by tossing a coin - heads for left, tails for right. Two additional items were chosen to ensure that the children understood the instructions and then these were inserted just over halfway through. In the event these were not really needed. All the children seemed to understand the

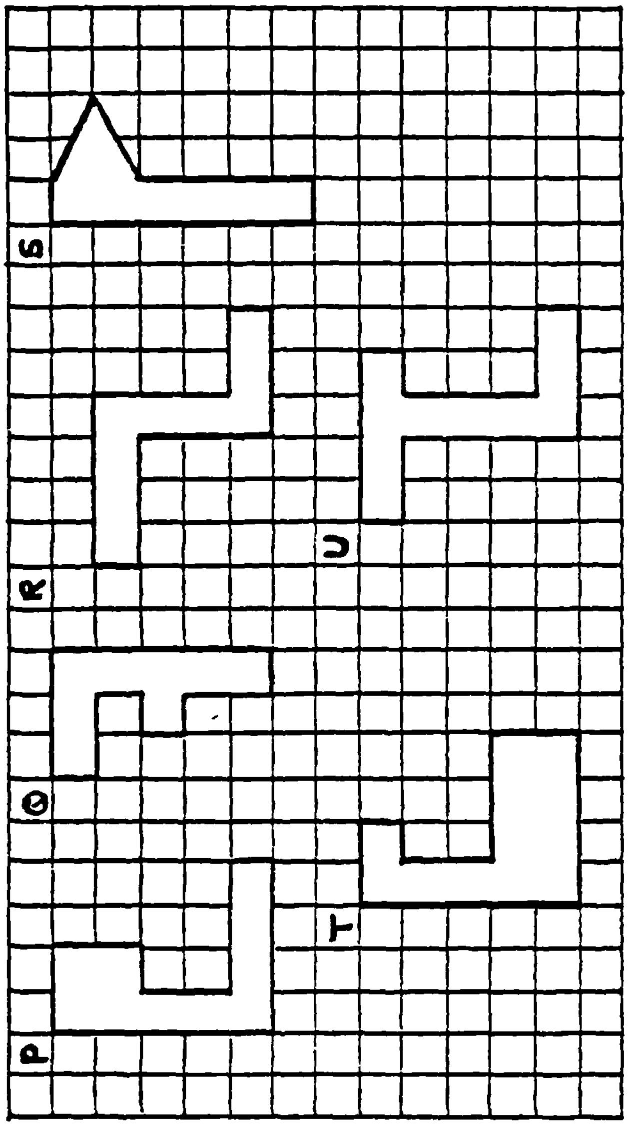


Figure 20 Six closed shapes used in preference alternative research

SOME QUESTIONS AJS A 40

Name Date

A -/- B/G A B C D

In each box you will find three figures. One at the top and two below. Decide which of the two below is most like the one above. Put a tick ✓ by this figure.

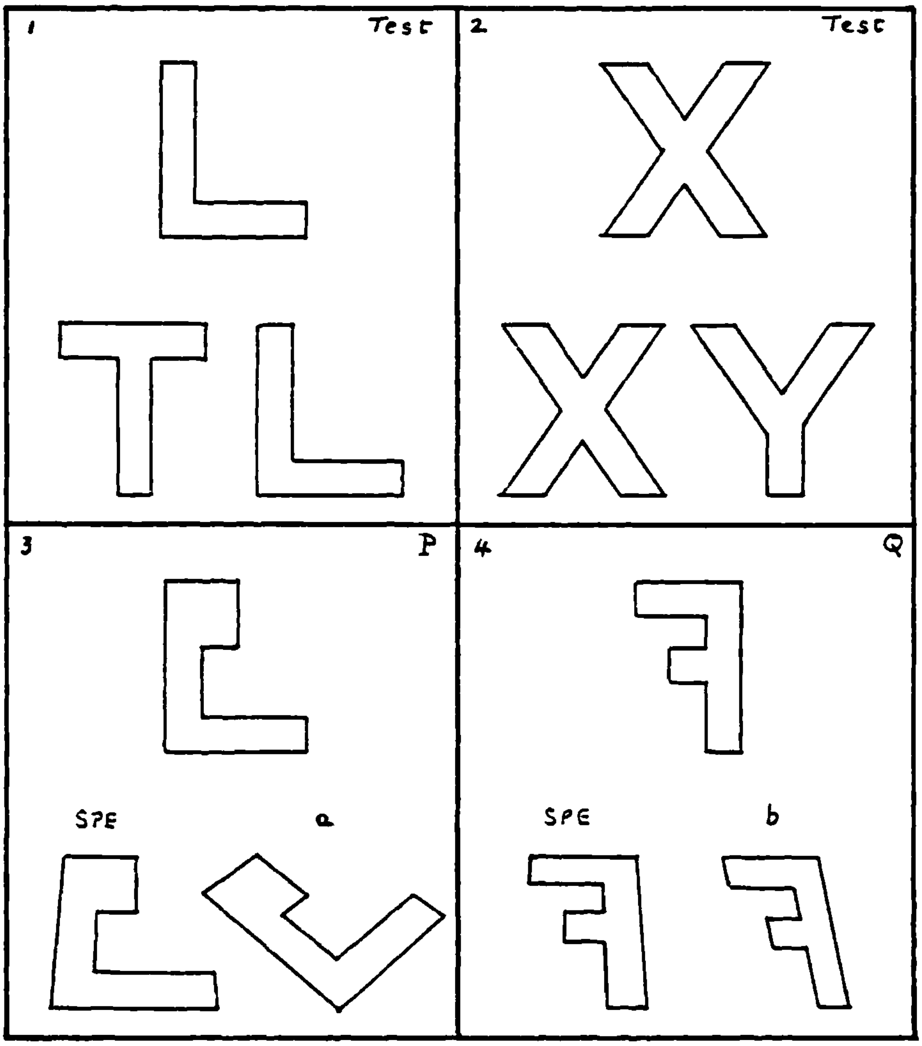


Figure 21 Forty alternative preferences (items 1-4)

SOME QUESTIONS AJS A 41

Name Date

A -/- B/G A B C D

In each box you will find three figures. One at the top and two below. Decide which of the two below is most like the one above. Put a tick ✓ by this figure.

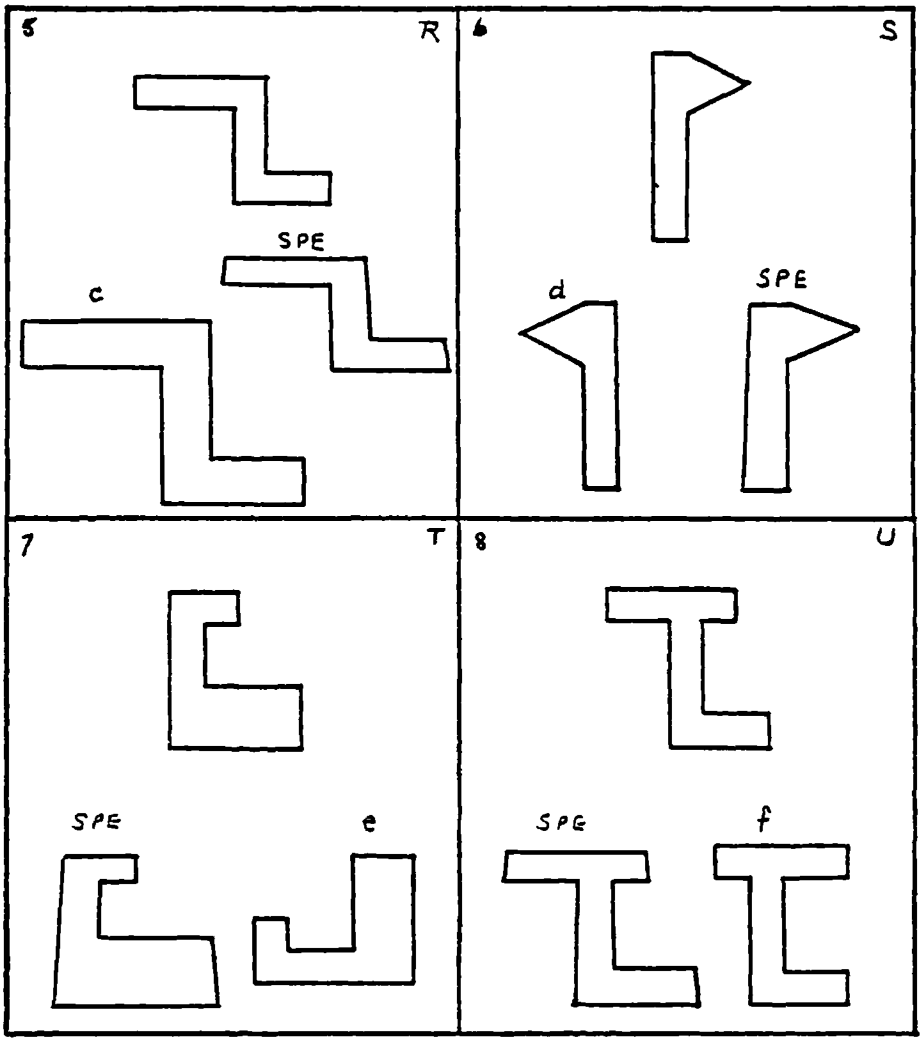


Figure 21 continued (items 5-8)

SOME QUESTIONS AJS A 42

Name Date

A -/- B/G A B C D

In each box you will find three figures. One at the top and two below. Decide which of the two below is most like the one above. Put a tick ✓ by this figure.

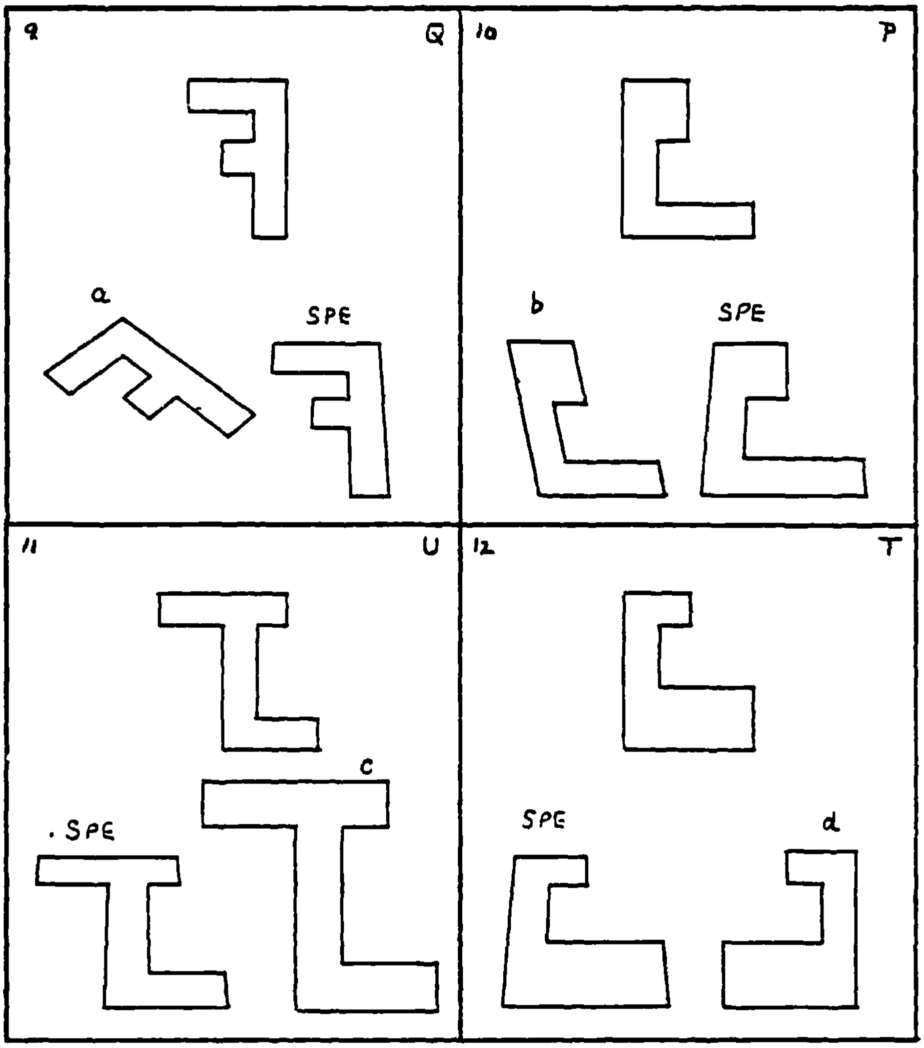


Figure 21 continued (items 9-12)

SOME QUESTIONS AJS A 43

Name Date

A -/- B/G A B C D

In each box you will find three figures. One at the top and two below. Decide which of the two below is most like the one above. Put a tick ✓ by this figure.

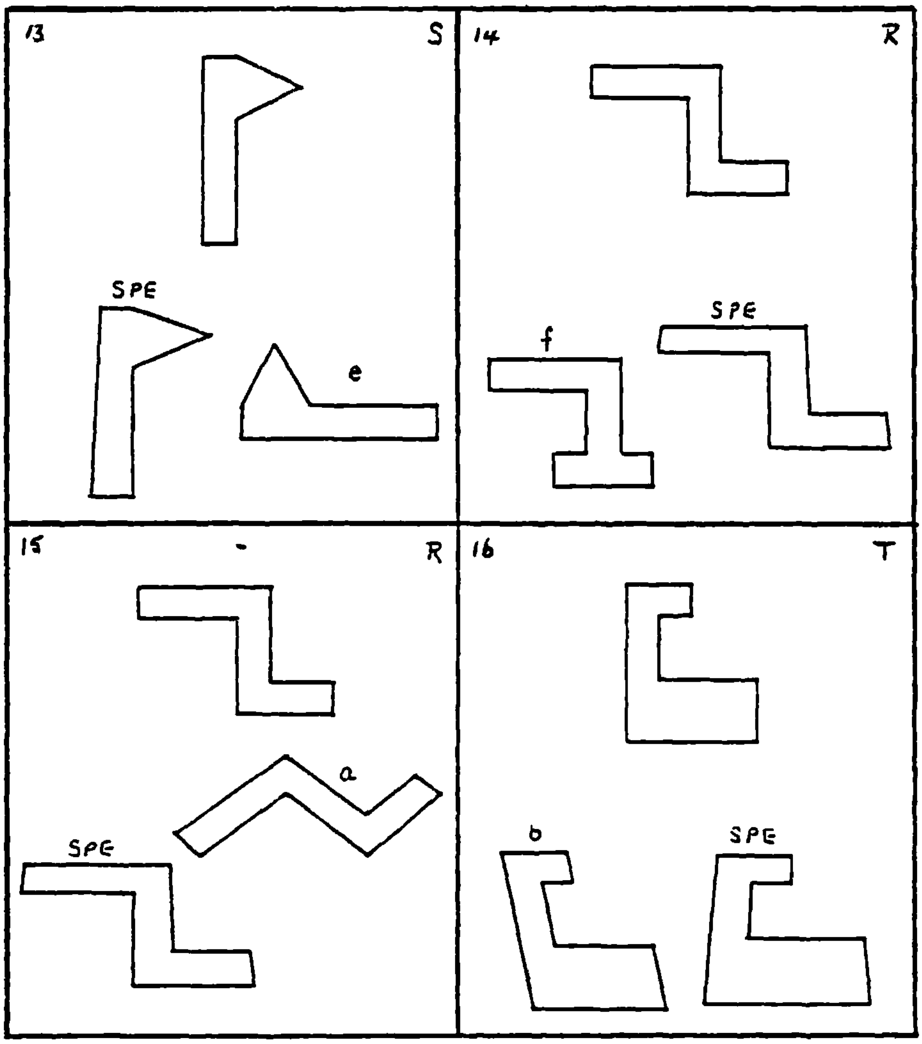


Figure 21 continued (items 13-16)

SOME QUESTIONS AJS A 44

Name Date

A -/- B/G A B C D

In each box you will find three figures. One at the top and two below. Decide which of the two below is most like the one above. Put a tick ✓ by this figure.

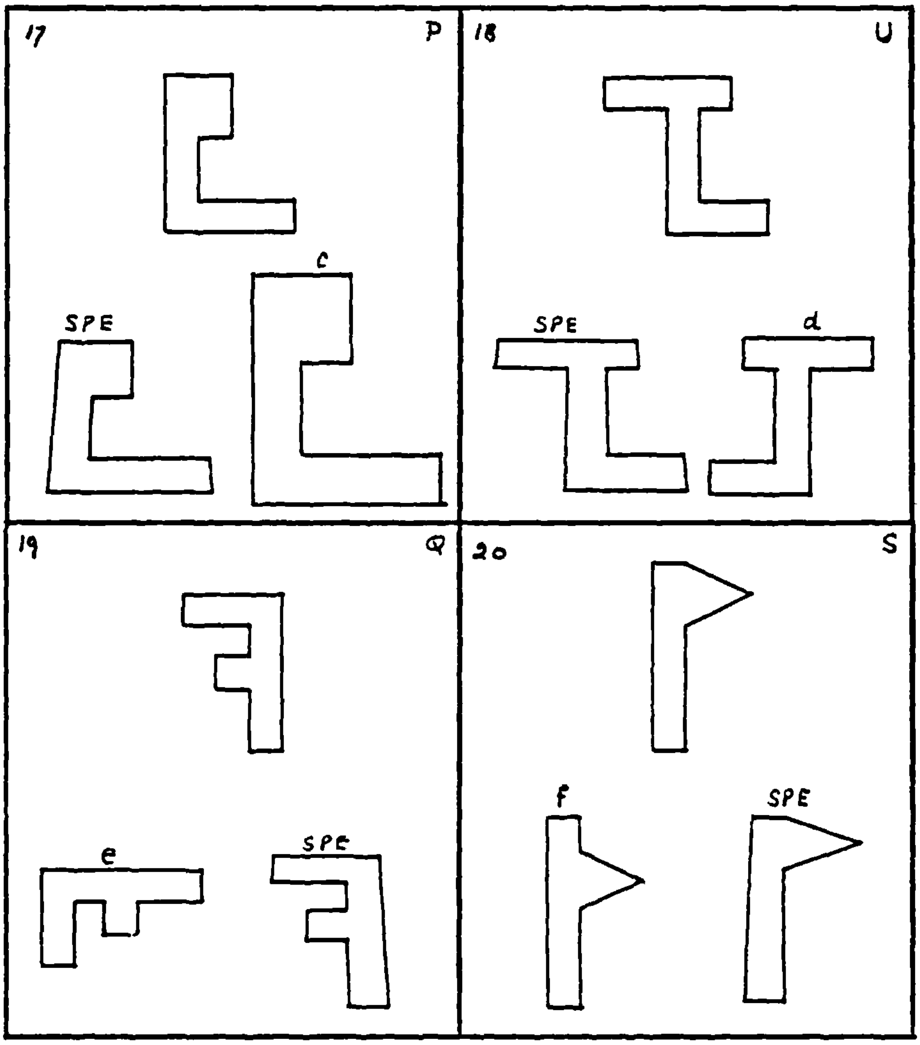


Figure 21 continued (items 17-20)

SOME QUESTIONS AJS A 45

Name Date

A -/- B/G A B C D

In each box you will find three figures. One at the top and two below. Decide which of the two below is most like the one above. Put a tick ✓ by this figure.

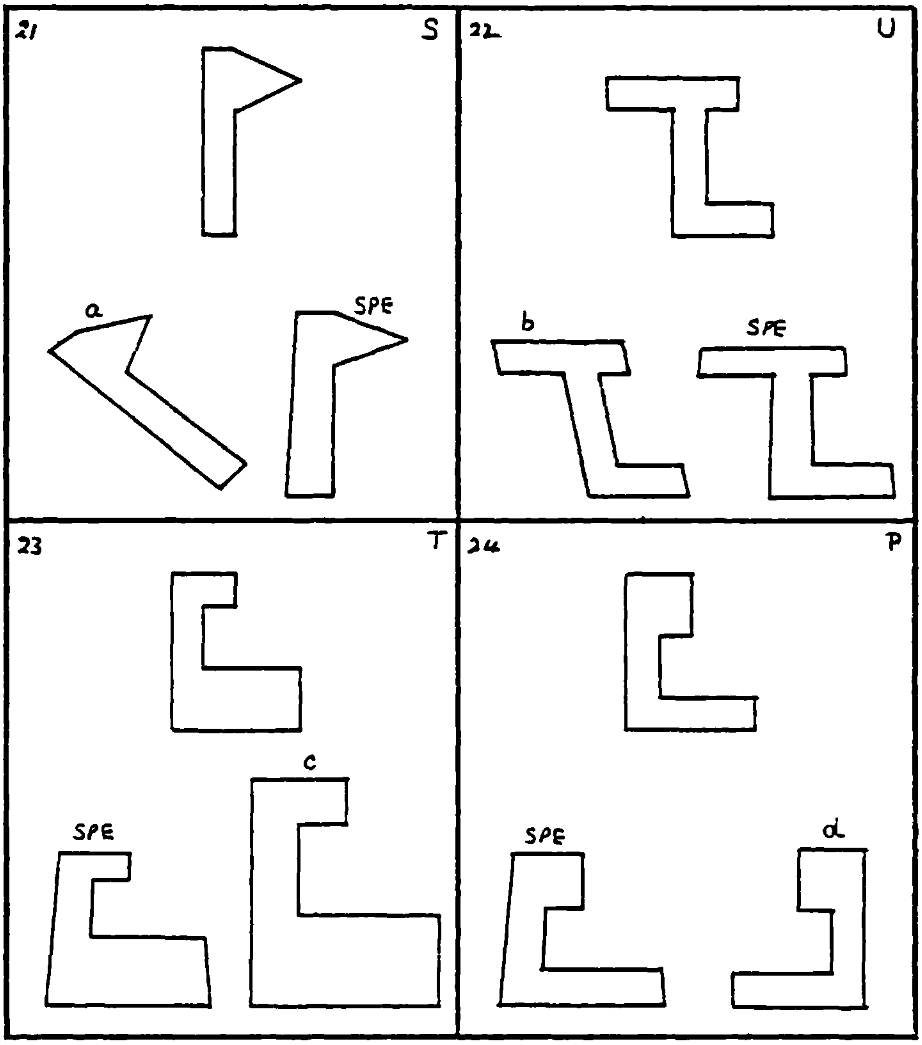


Figure 21 continued (items 21-24)

SOME QUESTIONS AJS A 46

Name Date

A -/- B/G A B C D

In each box you will find three figures. One at the top and two below. Decide which of the two below is most like the one above. Put a tick ✓ by this figure.

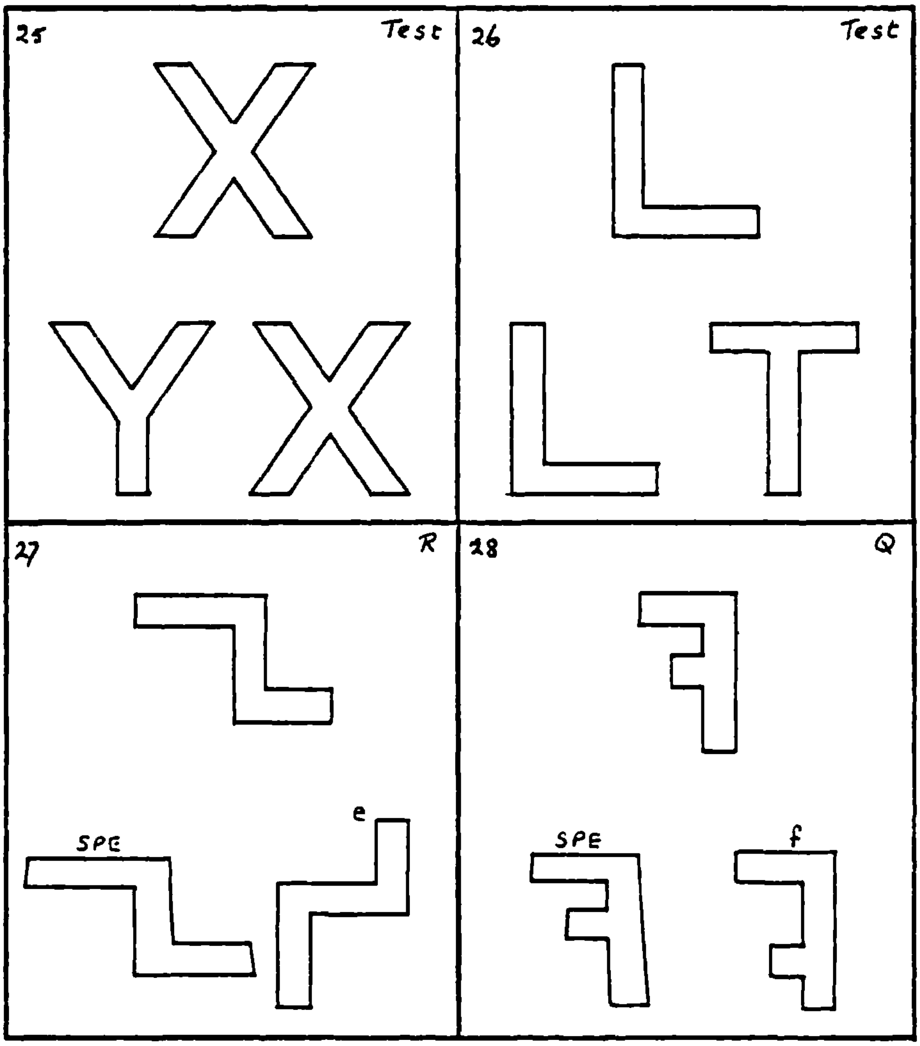


Figure 21 continued (items 25-28)

SOME QUESTIONS AJS A 47

Name Date

A -/- B/G A B C D

In each box you will find three figures. One at the top and two below. Decide which of the two below is most like the one above. Put a tick ✓ by this figure.

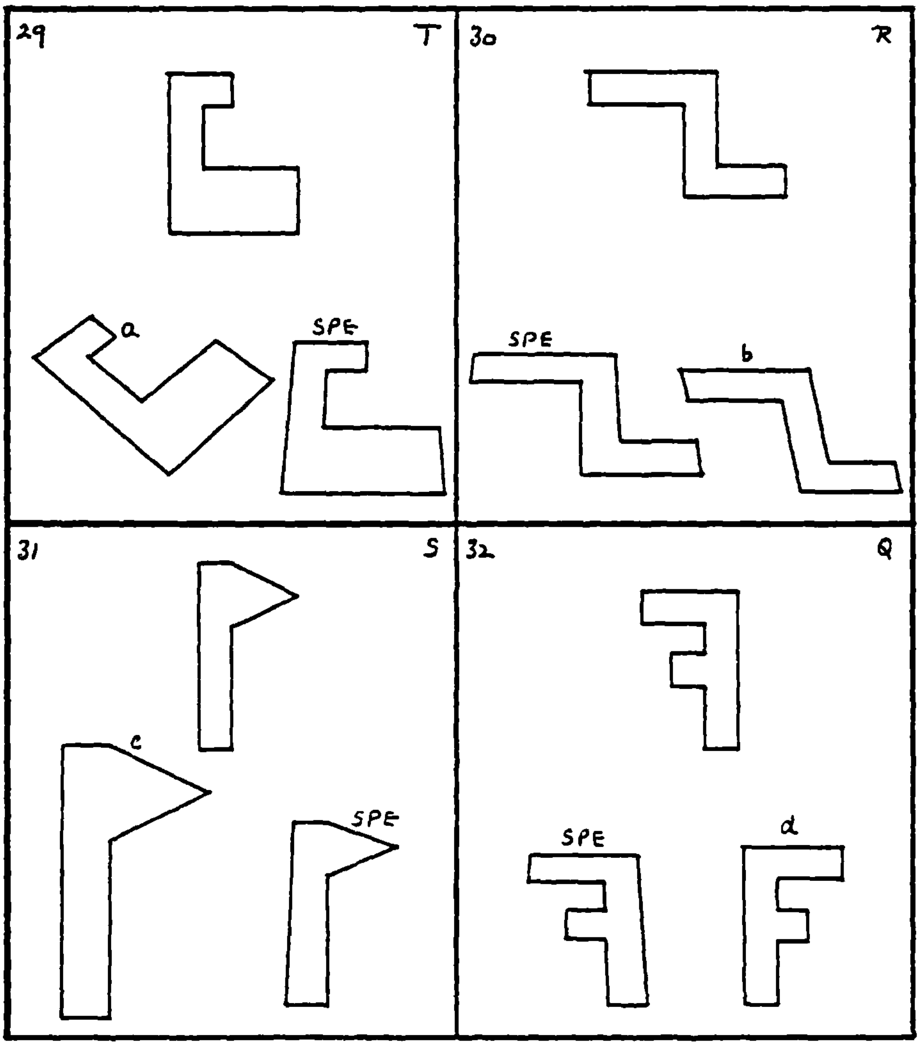


Figure 21 continued (items 29-32)

SOME QUESTIONS AJS A 48

Name Date

A -/- B/G A B C D

In each box you will find three figures. One at the top and two below. Decide which of the two below is most like the one above. Put a tick ✓ by this figure.

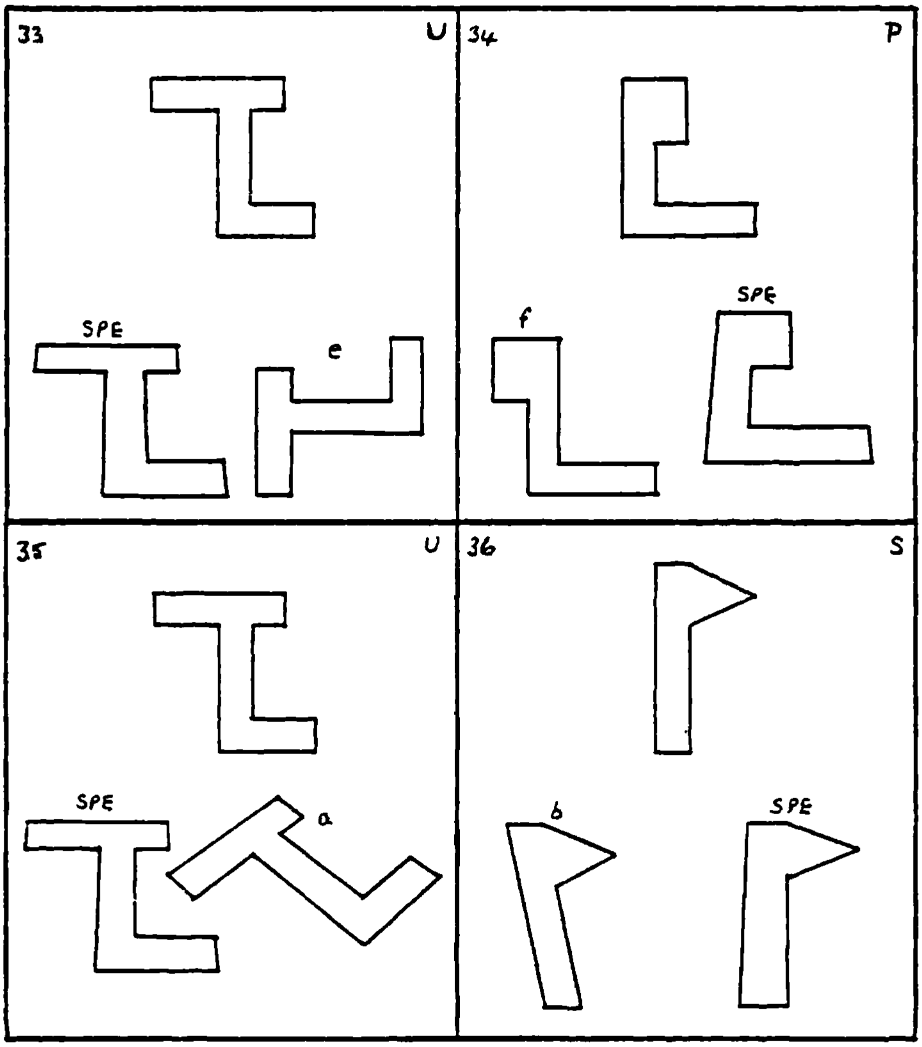


Figure 21 continued (items 33-36)

SOME QUESTIONS AJS A 49

Name Date

A -/- B/G A B C D

In each box you will find three figures. One at the top and two below. Decide which of the two below is most like the one above. Put a tick ✓ by this figure.

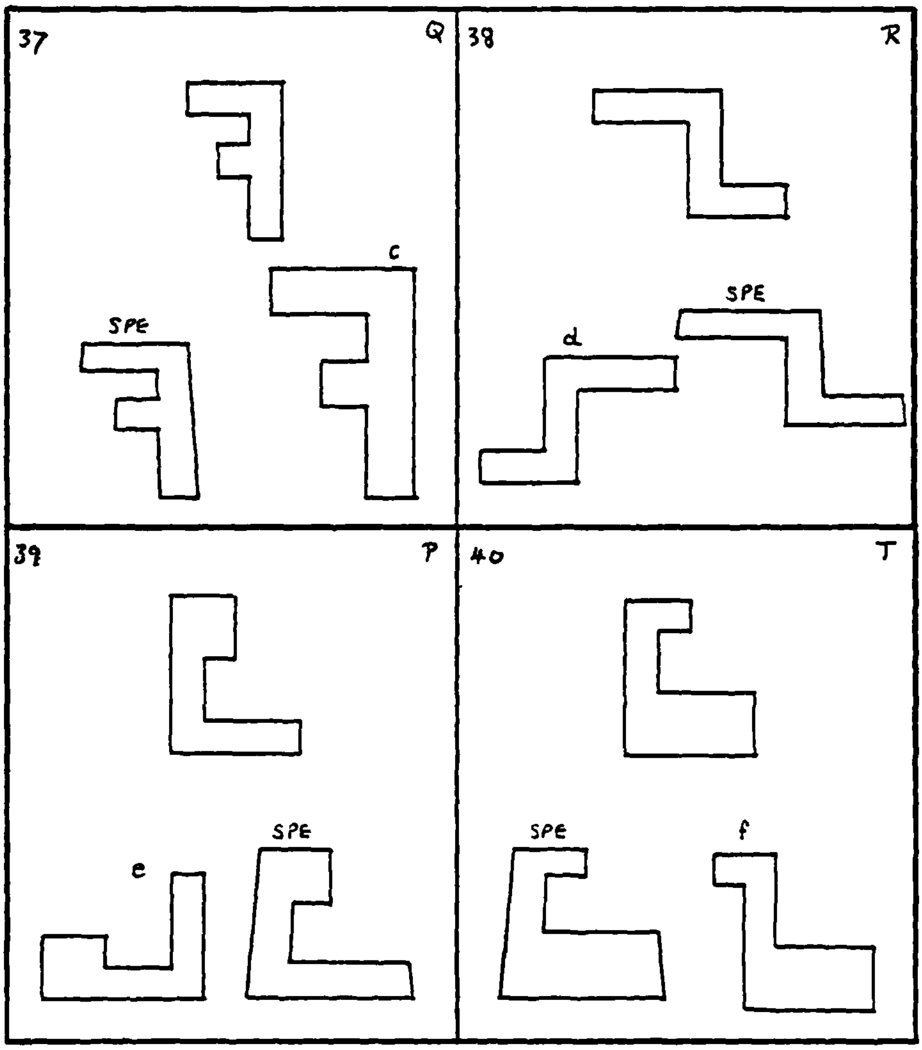


Figure 21 continued (items 37-40)

idea of choosing one or the other without the need for repetition of the instructions.

It follows then that the test of 40 questions consists of 4 questions put there to help prevent a misunderstanding in the instructions and that the remaining 36 separate into 6 each of (a) through to (e), there being P(a), Q(a), R(a), S(a), T(a) and U(a) in some order and similarly for (b) (c) (d) (e) and (f). These are given in A40 to A49; figures 20-29. The figures P through to U and the transformations are labelled for convenience but these labels were absent when the test was administered. As each of (a) to (f) has six questions each child has six scores (a) to (f) each of which is an integer 0 to 6 inclusive. A score of 6 shows a complete preference for the standard projective equivalent and a score of zero shows a complete preference for the alternative. A score of 3.0 shows no preference either way.

It should be mentioned that I interviewed all the children on an individual basis and that there was little chance of one child talking to another and hence influencing his choice. The forty questions were answered quite quickly by the children and it was unlikely that they would be able to remember or explain their methods of dealing with the choices.

A very great effort was made to be neutral. I did all I could to ensure that I did not influence children's choices which I accepted without change of voice or any signs, visual or verbal, of approval or disapproval. On occasions some children were asked to say why they had made a particular choice. On no occasion did I then change the choice which had already been made - even when one child said that she really meant the other choice.

Some children were quite incapable of saying why they had made a particular decision - but nevertheless appeared sure in their minds that the decision they had taken was the one they intended. Others would indicate that the one they had chosen was because it was "more like" the original - thus emphasising that they did understand and follow the original instructions. Still others indicated that they made their choices, at least on occasions, by a negative process. They mentioned particular aspects of a figure that prevented it from being a choice. Such phrases as "it's too large", "this part's been moved", "this bit's too low", "it's been turned", "it's thicker here and thinner here", "it's back to front" and so on were used to justify choosing the other figure which did not have these undesirable characteristics.

There were also some indications that each frame was considered on its merits independent of the previous ones.

A null hypothesis was chosen that there is no bias towards or away from the standard projective equivalent. The null hypothesis $H_0 : \bar{x} \neq 3.0$ was chosen. Using a t test at the 1% level the null hypothesis H_0 was replaced by $H_1 : \bar{x} > 3{\cdot}0$, that is there is a bias towards the standard projective equivalent. The results are contained in table 2.

Results

Table 2

Results of Preference Choices

Alternative Equivalent to S.P.E.		n	$\bar{x}$	σ	s	Σx	Σx^2	t	H_0 rejected?
Oblique Projective Equivalent	(a)	27	5.37	0.87	0.884	145	799	13.9	rejected
Affine Equivalent	(b)	27	3.52	1.62	1.649	95	405	1.64	not rejected
Similar	(c)	27	4.62	1.06	1.079	125	609	7.80	rejected
Reflective Equivalent	(d)	27	5.22	1.09	1.121	141	769	10.29	rejected
Rotational Equivalent	(e)	27	4.67	0.72	0.734	126	602	11.82	rejected
Piecewise Congruent	(f)	27	3.93	0.66	0.675	106	428	7.16	rejected

With 26 degrees of freedom the 1% level t statistic is 2.779

Thus H_0 is rejected in all cases except (b) where t = 1.64 in the affine case

These results bear out the suggestions made by Geeslin and Shar (1979) that rotations and dilations (similarity) should be considered as more complex distortions. Piecewise congruent distortions and reflections also appear to fall in the same category.

The (a) experiments showed a very strong preference for the standard projective equivalent when compared with the oblique projective equivalent. This is not at all surprising. The idea of a different viewpoint is one that Piaget suggests is very difficult to handle. This will be especially the case when the problem of the viewpoint was implicit rather than explicit. A mean of over 5 is to be expected.

The lack of bias one way or the other and the acceptance of the null hypothesis in the affine case needs further consideration. There is a "sameness" about the two figures which can be attributed to their having very similar amounts of distortion from the original, both are orientated the same way, so again the research findings of Geeslin and Shar are not irrelevant. But of course there are essential differences between the two figures. One of the questions that arises is whether the children took any notice of the parallelity of the lines in the affine case. It appears that children were not seeing the parallelity as being "more like" the parallelity of the original figures. Although it is not properly part of this present study the introduction to the concept of parallel lines is worthy of some consideration.

A further experiment. Some concern was felt that each pair of alternatives contained a direct projective transformation and that such a presence in each choice may have contributed to the preference for the projective choice. To determine whether this was the case or not, a further set of eight choices was made. Here a rotational transformation was given against two each of affine, piecewise continuous, similar and direct projective. It was thought that the rotational figure was the one most likely to produce a mental set. The figures used were the figures already met labelled P, Q, R and T. S was omitted as being rather unlike the others being easily recognised and labelled as a flag and U was omitted to prevent the experiment being too long for the children. The left or right positions of the rotation were again chosen by the toss of a coin. The eight choices are given in figure 22.

In the analysis of these data given in table 3, two scores were made for each child. (i) the number of times a rotational choice was made on the first six and then (ii) the number of times a rotational choice was made on the last two.

SOME QUESTIONS AJS A 50

Name Date

A -/- B/G A B C D

In each box you will find three figures. One at the top and two below. Decide which of the two below is most like the one above. Put a tick ✓ by this figure.

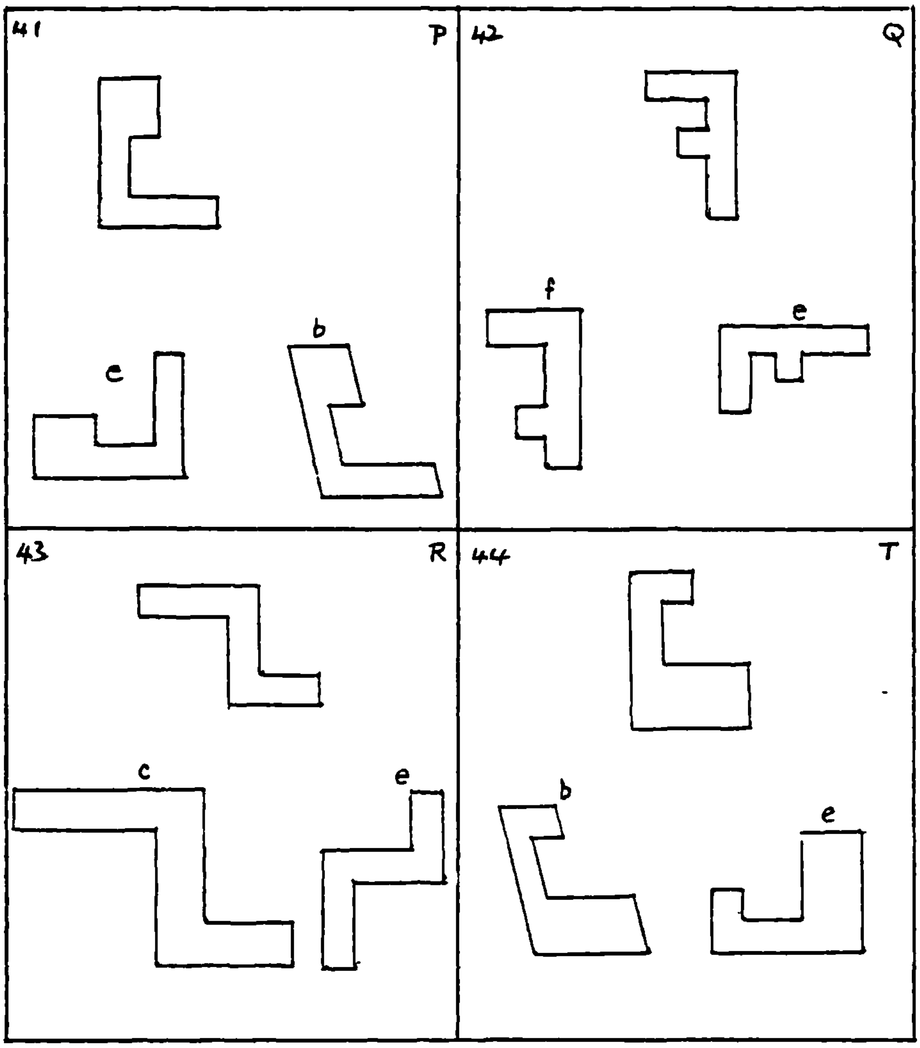


Figure 22 Eight further alternative preferences (items 41-44)

SOME QUESTIONS AJS A 51

Name Date

A -/- B/G A B C D

In each box you will find three figures. One at the top and two below. Decide which of the two below is most like the one above. Put a tick ✓ by this figure.

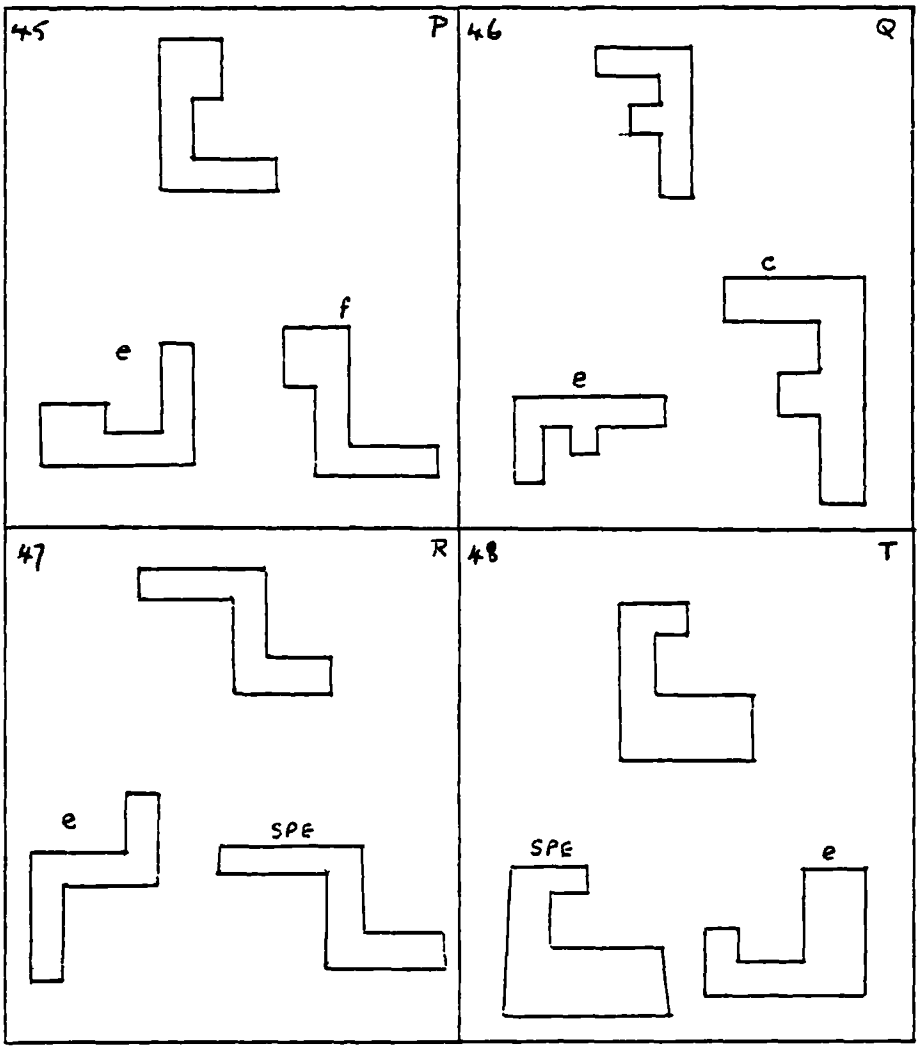


Figure 22 continued Eight further alternative preferences (items 45-48)

Table 3

Results of Further Preference Choices

(i) rotational choice first six (Max 6)	(ii) rotational choice last two (Max 2)	frequency
6	2	5
5	2	3
4	-	0
3	0	2
2	2	1
2	0	3
1	0	9
0	0	4
		27

These figures show that the children were being quite consistent. On the last two every child chose either both rotational equivalents (one being on the left and the other on the right) or both standard projective equivalents. No child chose one of one and one of the other. Those children choosing5 or 6 rotational choices on the first six (eight of them) were from the top three quarters of the ability range. Hence there is a tendency having chosen a rotation to do so when there is a projective alternative. There is also a tendency very obviously working in the other direction. Not having chosen the rotation on the first six they, on the whole, continued not to do so. As mentioned previously the children appeared to be treating each frame on its merits regardless of what had happened previously. A few of the children's comments made at the time suggest that this was so.

What can be stated quite conclusively is that the presence of the rotational choice in every frame did not of itself lead towards it being the selected choice. In spite of the common element of a rotational equivalent two thirds of the children rejected this choice when a standard perspective equivalent was available. It may then be asserted that the mere presence of the standard projective equivalent in the original test of thirty six items would not have contributed much to its selection in these cases either. Neither would it explain the acceptance of the null hypothesis in the affine equivalent case. It may be stated that children have a strong preference for a projective transformation rather than certain euclidean transformations; but this strong preference does not exist when the choice is between an affine and a projective transformation.

CHAPTER FOUR

The Development of a Projective Geometry Curriculum in the Primary School

Setting up the Research

Introduction. This chapter describes how a hunch was turned into a reality. I have, for many years, held a belief that projective geometrical concepts could prove valuable in primary schools. A hypothesis providing a basis for this study is that children who have experienced implicit or explicit learning activities in projective geometry will have developed a better facility in dealing with line drawings, being able to produce, use, and interpret line drawings in a variety of contexts. Productive learning activities provide opportunities for development of logical and psychological structures, which in turn allow for intellectual and conceptual clarity as a basis for acquiring concepts in primary geometry. They also integrate, it is suggested, with other subject areas making geometry more relevant and vital to children.

Thus the main aim is to provide valuable and interesting opportunities for children to be actively involved in implicit and explicit projective experiences.

Further aims are to make projective links with other subject areas and to provide teachers with a conceptual framework which will enable them to operate effectively and with confidence.

The main function of the materials is to equip teachers with a curriculum resources package which will detail projective activities in such a way as to allow their use in a variety of circumstances. Another function is to provide feedback information for the teacher about each individual child's thoughts in perception and visualisation.

Initially it was conceived that the target audience was seven to nine year olds, but as the study developed it was extended up to eleven, and some of the exercises, it may be suggested, are not inappropriate for the lower secondary school.

The work extended over six school terms. Normally it involved a visit to the school once a week for a period of two hours, although variations did occur from time to time.

As the investigation was opening up new fields procedures used in the implementation of the study were to some extent tentative and speculative. For example, in the projective area there are few examples of children's work which are suitable for use in primary schools, and so there are few directly relevant indications of appropriate methods of operation.

Some pilot tests were constructed for use in the age range seven to nine years old. From a review of research literature it would seem that this was a reasonable age for a trial scheme. An evaluation of the pilot tests is given later in this chapter. The experience gained in administering the pilot tests provided pointers to a further delving into research literature in areas which had not been expected to be directly relevant. Such a reading of further literature determined further tests of a pilot nature - with more questions being posed than answered - and with a greater degree of classroom observation being required than was originally envisaged. It was found, that the majority of the researchers in other but connected areas of interest, professed or exhibited little knowledge of mathematical implications of pure projective geometry. Often a euclidean paradigm was evident and so it was difficult to interpret the results within a modern framework such as that provided by the Erlangen programme or a Piagetian version of it. Findings tend also to show that patterns of children's thinking are complicated and confused.

It soon became evident that the research design would not be amenable to a classical evaluation. During the main research, the scope of the study expanded and there were constant changes which affected the evaluation procedures which were both formative and summative.

The initial pilot tests. These were discussed in chapter three. The very early attempts to modify a Piagetian test and to quantify the results is documented as are the pilot tests on interpretations of line drawings. Such results were used to modify the curriculum package in projective ideas.

The pilot test had its origins in a previous study as a requirement of an M. Ed degree. A series of projective activities in a work card format was produced by the investigator in 1977. This followed the lines of Dienes and Golding, (1967), though they were considerably modified to emphasise projective rather than affine or topological ideas. The work cards provided a starting point but it soon became evident that the programme was inadequate. Part of the problem was the ineffectiveness of a simplistic Erlangen approach.

Further researchers, Martin (1976a, b) in particular, stress the need for more research in the area as it is probable that topological, projective, affine, and euclidean concepts may all be suitable at various and different stages of development.

Giles (1979) uses isometric dotty paper for affine transformations to and from three dimensional solids and their two dimensional representations. This suggests that there could be a similar projective activity using projective dotty paper.

It seems that, from the literature which is available about primary geometry, there is much turmoil and confusion about children's thinking.

At one time in the research, in fact, the investigator was coming to the conclusion that children learn geometry in very personalised and idiosyncratic ways with very little perceivable structure. This was especially reinforced by Donaldson (1978).

Hence with little or no structural preconceptions, the original series of simplistic Erlangen work cards in projective ideas were modified and expanded. The work cards so formed were essentially experimental, combining ideas from a variety of sources including other subjects. Originally the series was introduced rather tentatively in some teachers' notes which allowed a "pick and mix" approach in the selection of appropriate work for children. There was an implication that children were probably best able to choose activities which they found intrinsically interesting. Nevertheless some of the cards were sequenced where cumulative learning was necessary. This was especially true of the straightness activities where concepts had to be developed in a hierarchical order. As the study progressed an attempt was made to introduce some rationale into an approach to the activities by the use of modes of geometrical learning. These are considered in detail in chapter five.

The Main Research : Aspects of Summative Evaluation

Changes in the research design. The initial pilot tests detailed in chapter three had a substantial influence on the development of the research and indeed on the research design as well. Not the least of these was the effect of the pilot scheme on the class which was used for it. It was evident from discussions with the class teacher and with children that they were beginning to switch on to my (projective) wave length. This caused doubts to arise about the validity of the summative evaluation, a pretest and post-test with a proposed experimental and control group. Due to syllabus constraints, the class teacher of the pilot class had begun work on maps prior to my production of a projective approach to the same subject. Thus it became obvious that the pilot scheme class could not be used in the main research except for pilot trials with new research materials.

Partly in response to this dilemma and partly to use a teacher who could come to the research programme with no previous experience of my thinking, it was decided to use a different class for the main research. From the experience gained with the pilot scheme class it was clear that a greater amount of informative feedback should be forthcoming with an older age range. In consultation with the teacher and the headmistress, an experimental group, class S consisting of 36 mixed ability ten year old children was chosen. By this time a good proportion, about two thirds, of the projective activities were in a form ready to give to a teacher. It was decided initially that the teaching should be carried out by the class teacher rather than the investigator.

Two parallel classes to the experimental class (class S) were available as controls. Class T with 35 children which was chosen as the control and class U with 36 children, which was not.

Summative evaluation. Before the main teaching began, four classes were given a pretest. These were the pilot class B and its control class A, the experimental class S and its control class T. The pretest consisted of a subset of the, as yet, incomplete set of activities. These were chosen for a variety of reasons. The overriding concern was to ensure that the control classes, in particular, did not have an unnecessarily long test session. Other points taken into consideration were the need: (a) to complement the initial pilot tests; (b) to give opportunities for children to respond in a variety of ways such as multichoice selection, free drawing, structured drawing, responses to visual illusion; and (c) to contain some activities which eight year olds could manage - a device used to help prevent some children giving up early on the test and thereby giving a minimal response to later items. It was realised at the time that this may make parts of the pretest rather trivial for ten year olds. To ensure reliability the test was administered to each of the four classes A, B, S, T, by the investigator. Each instruction was read and re-read once and efforts made to use exactly the same form of words with each class. Each class took approximately forty-five minutes to complete the test - though some children were anxious to have extra time, which was not allowed, to complete their free and structured drawings.

The pretest and post-test assignments (items 1 to 27) are given in figure 23. They are reduced to 75% of their original size. These are immediately followed by table 4 which is a tabulated form of the instructions actually used with the figures.

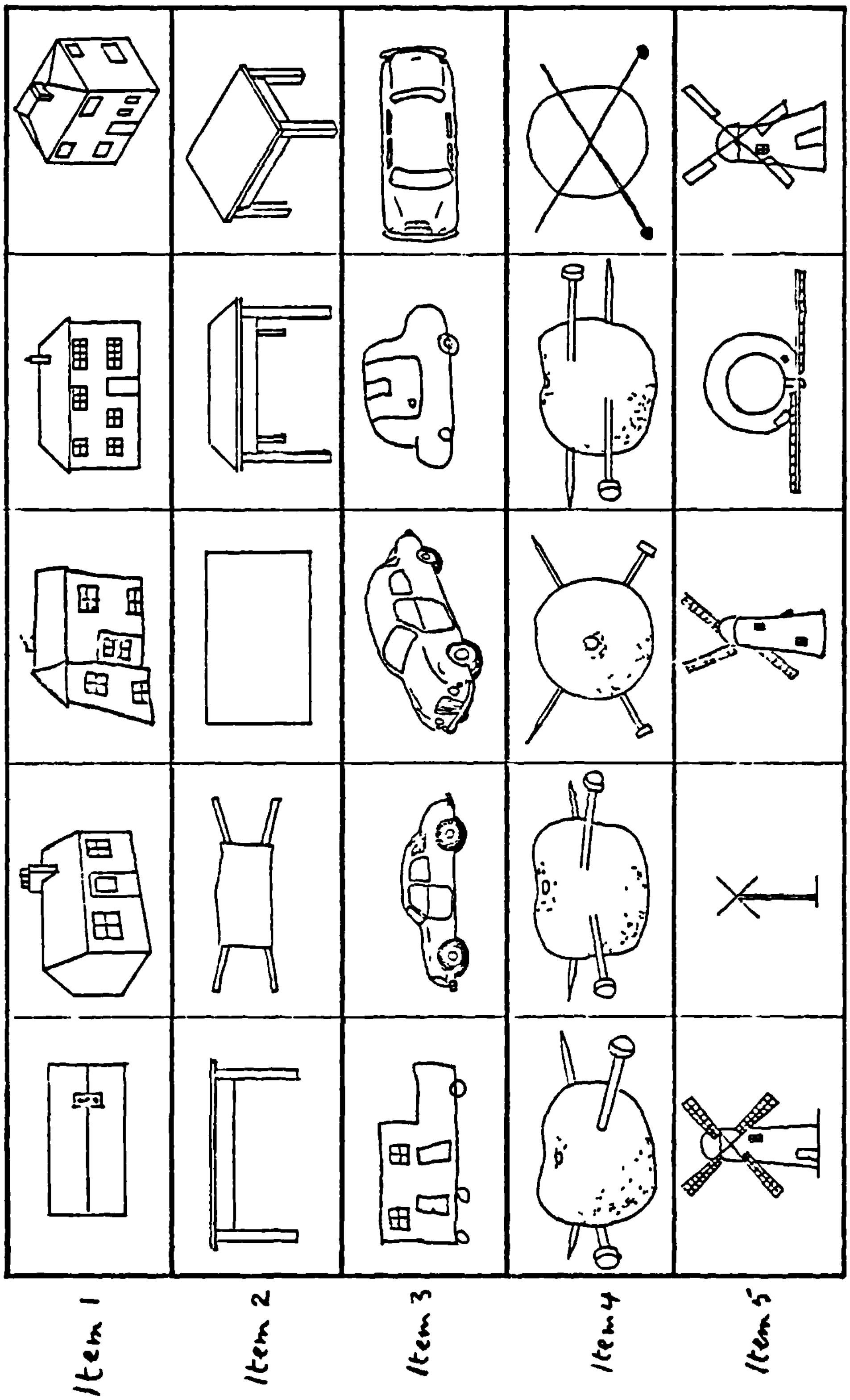
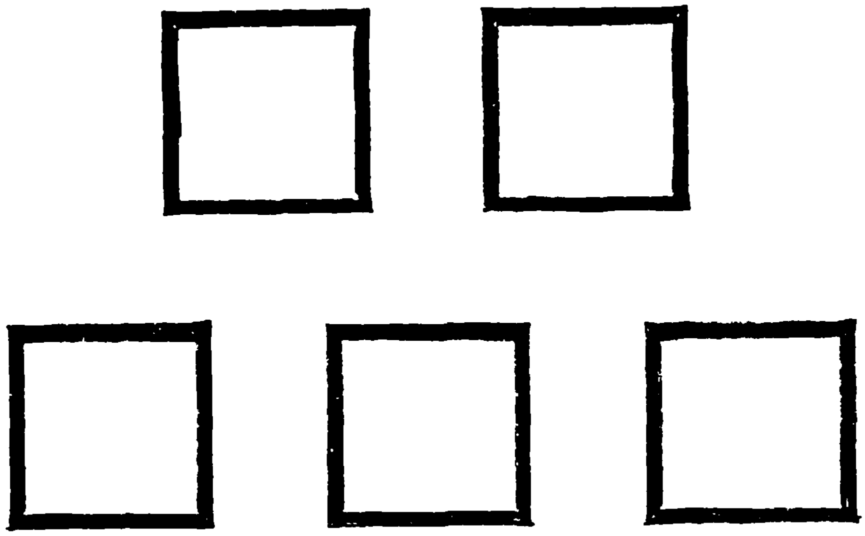
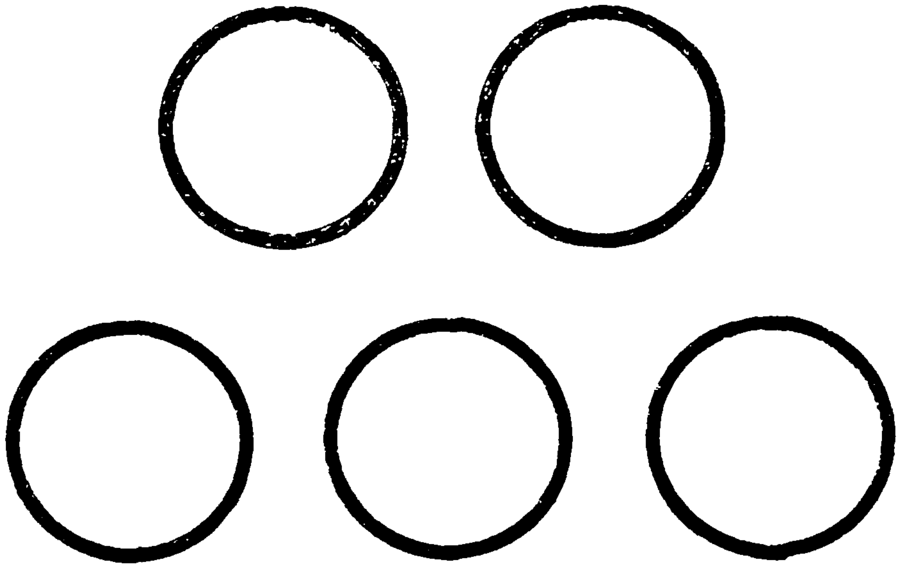


Figure 23. Pretest and post-test assignments (Items 1-5)

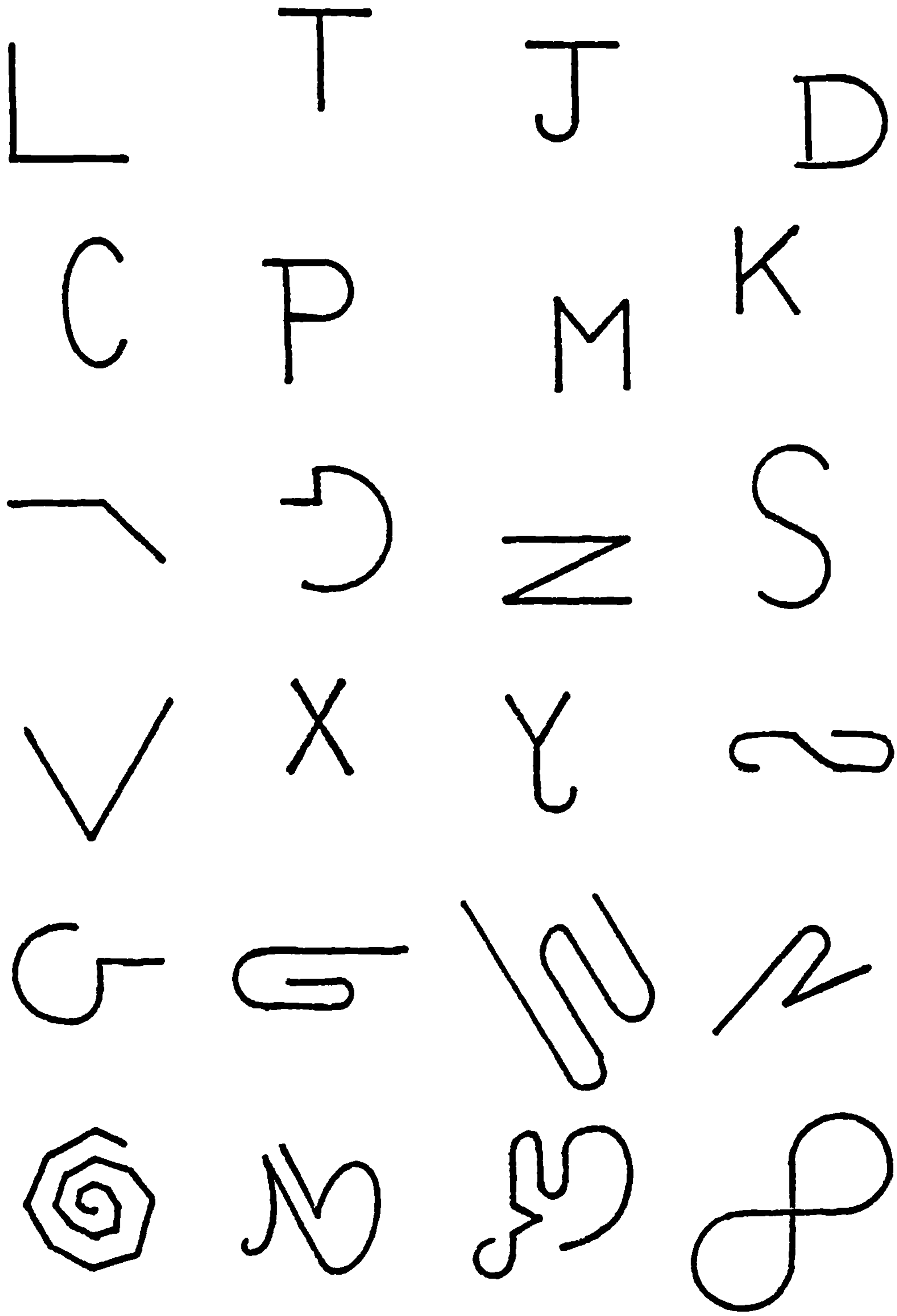


Item 6



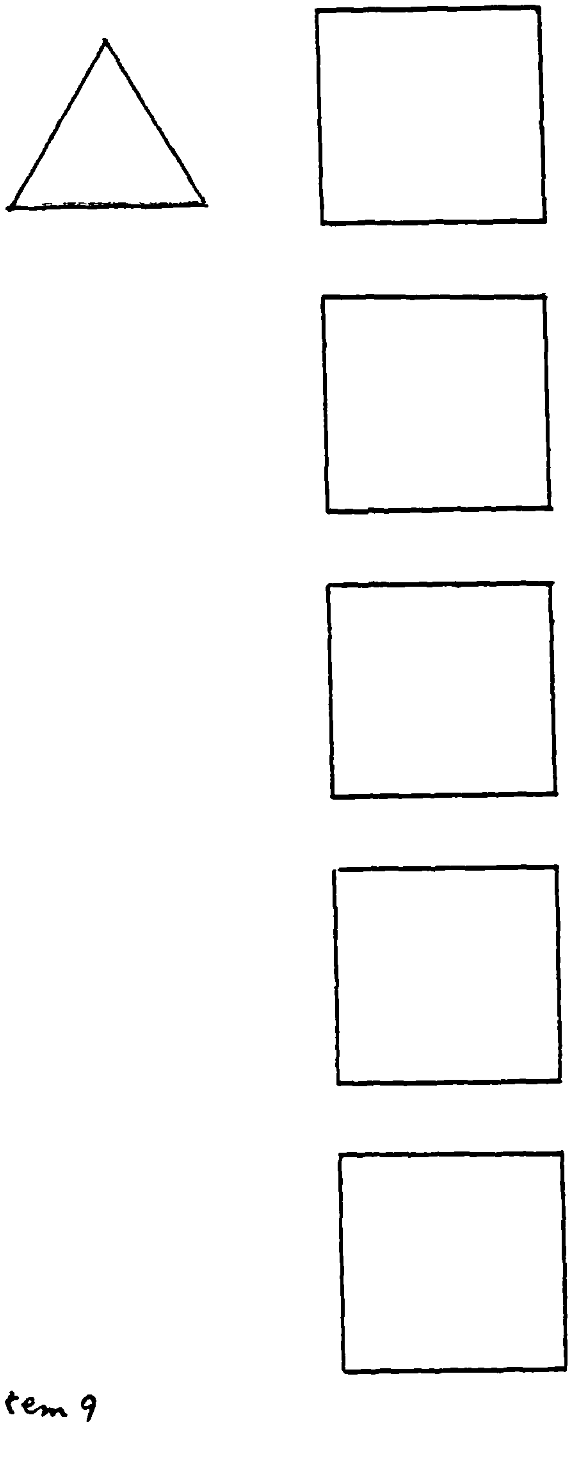
Item 7

Figure 23. Pretest and post-test assignments (Items 6, 7)



Item 8

Figure 23. Pretest and post-test assignments (Item 8)



Item 9

Figure 23. Pretest and post-test assignments (Item 9)

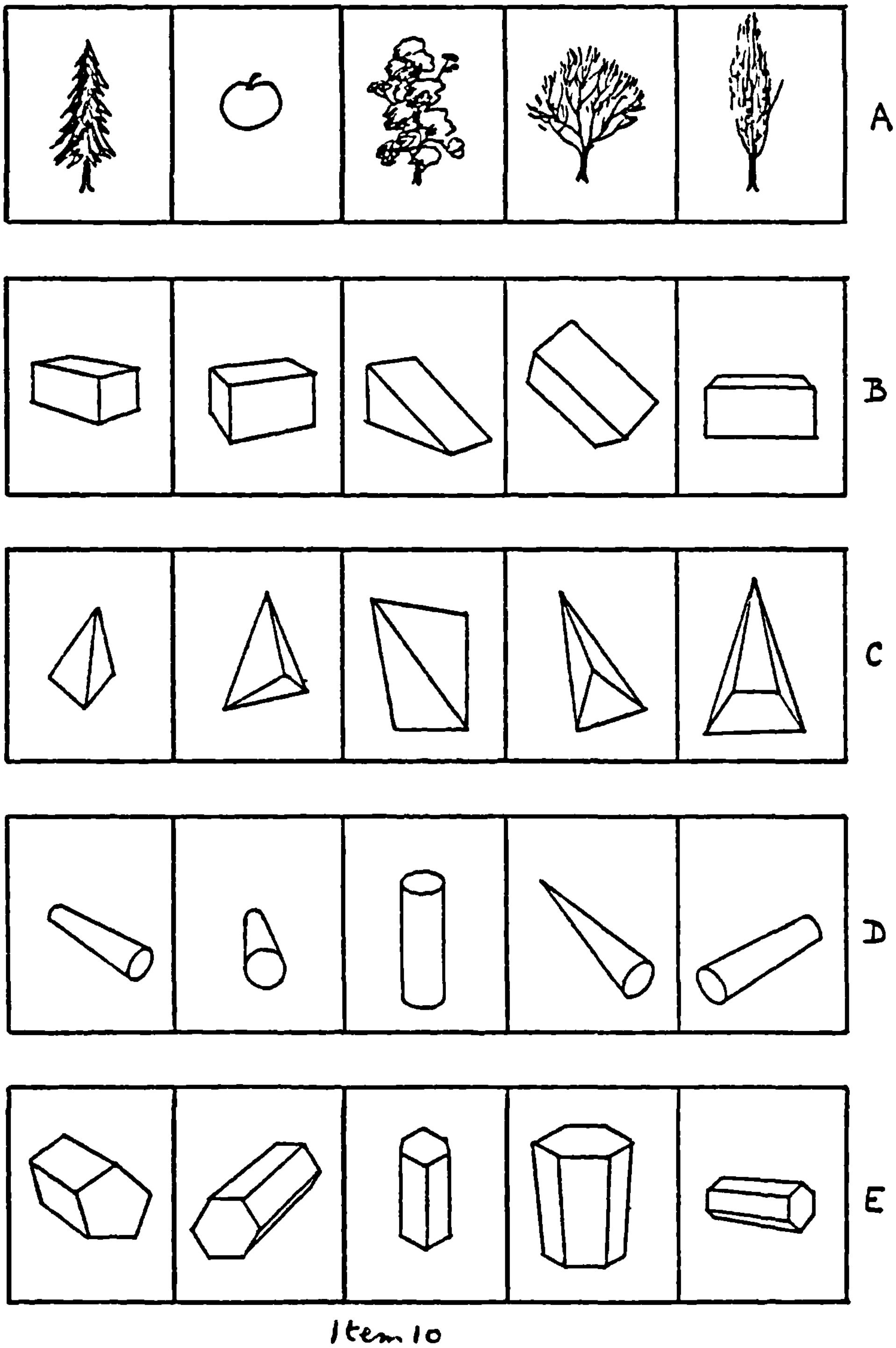
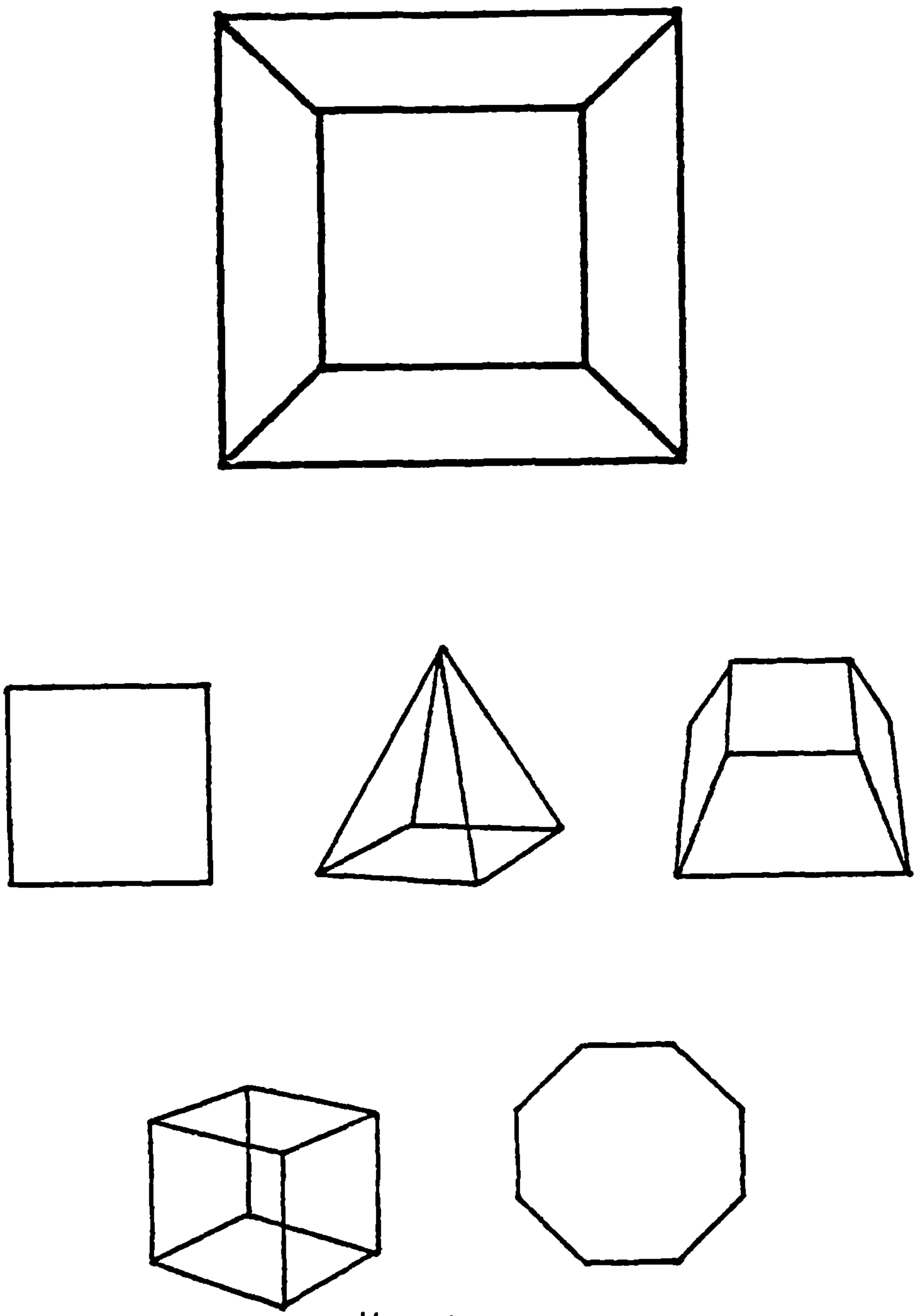


Figure 23. Pretest and post-test assignments (Item 10)



Item 11

Figure 23. Pretest and post-test assignments (Item 11)

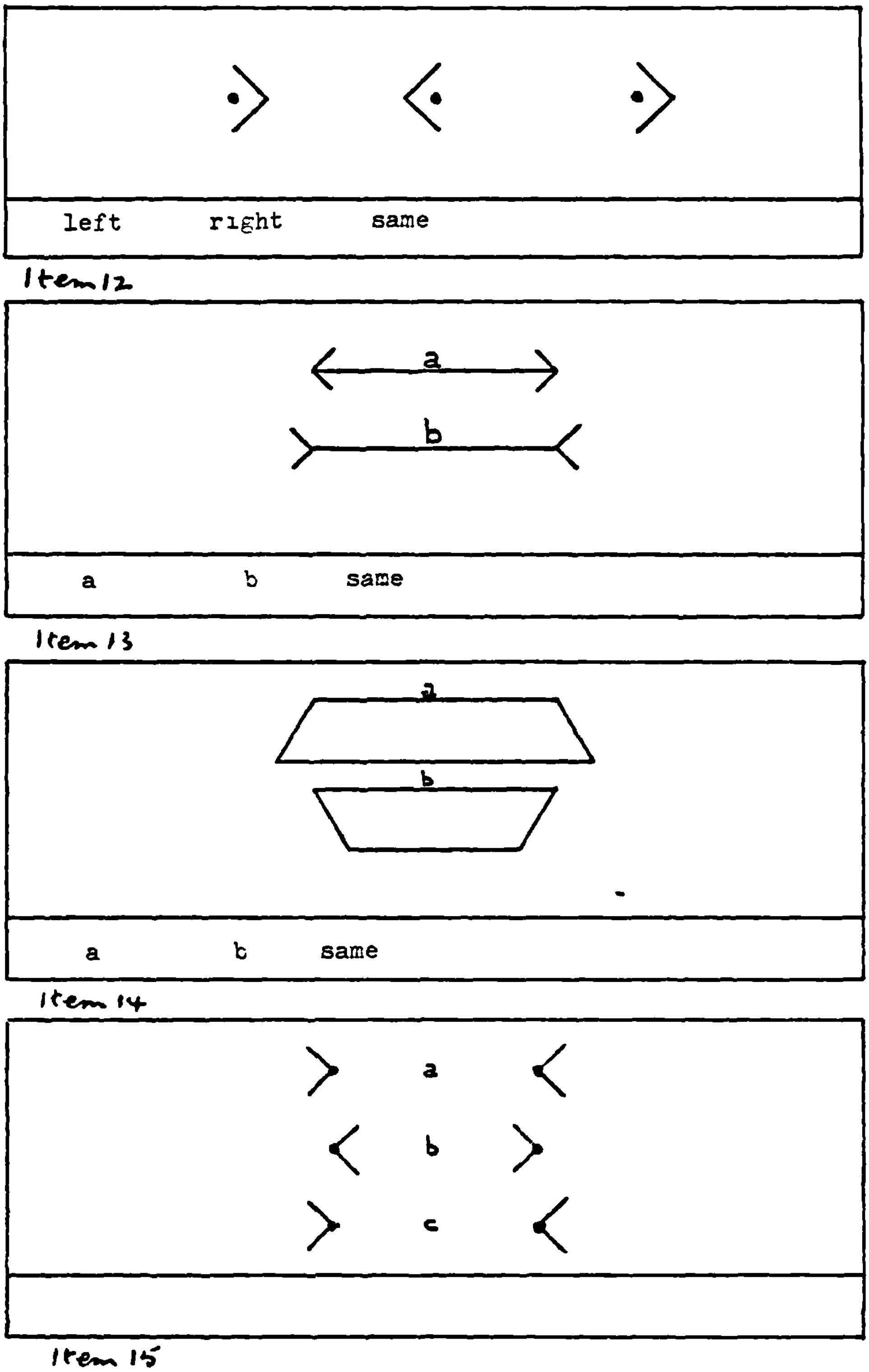
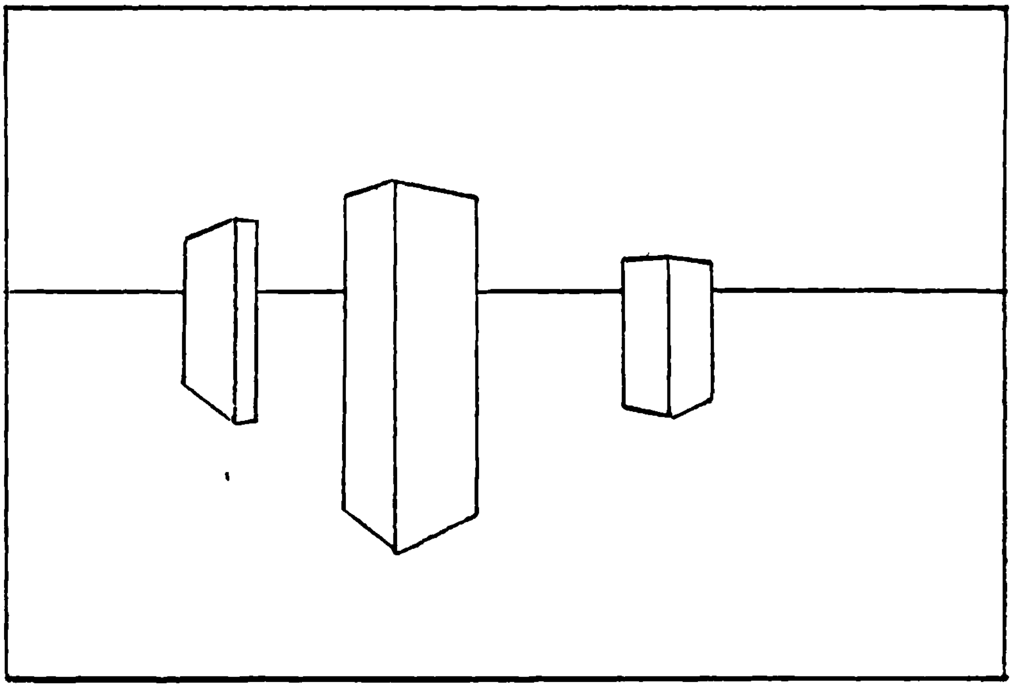
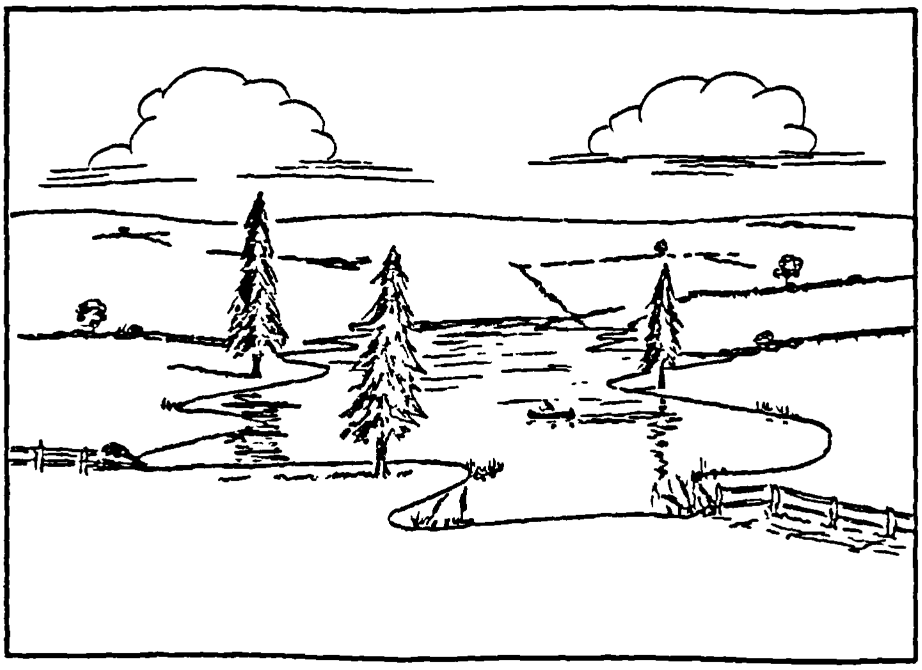


Figure 23. Pretest and post-test assignments (Items 12-15)



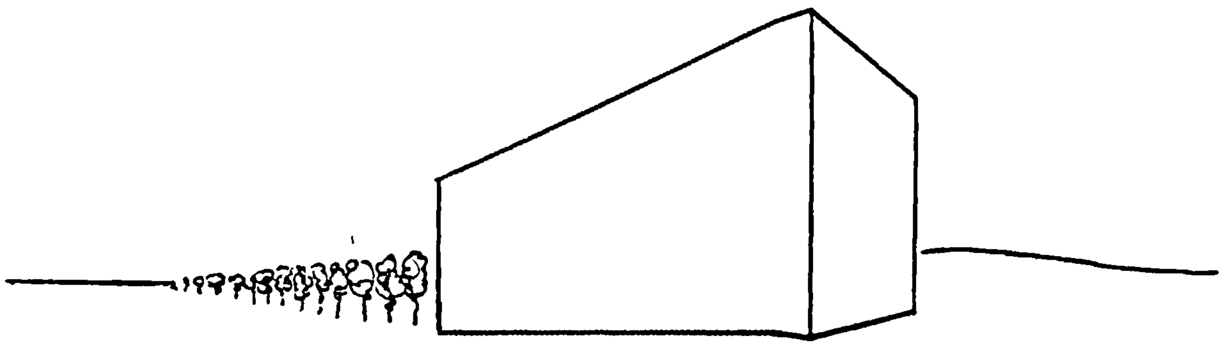
Items 16, 17.

Figure 23. Pretest assignments (Items 16, 17)



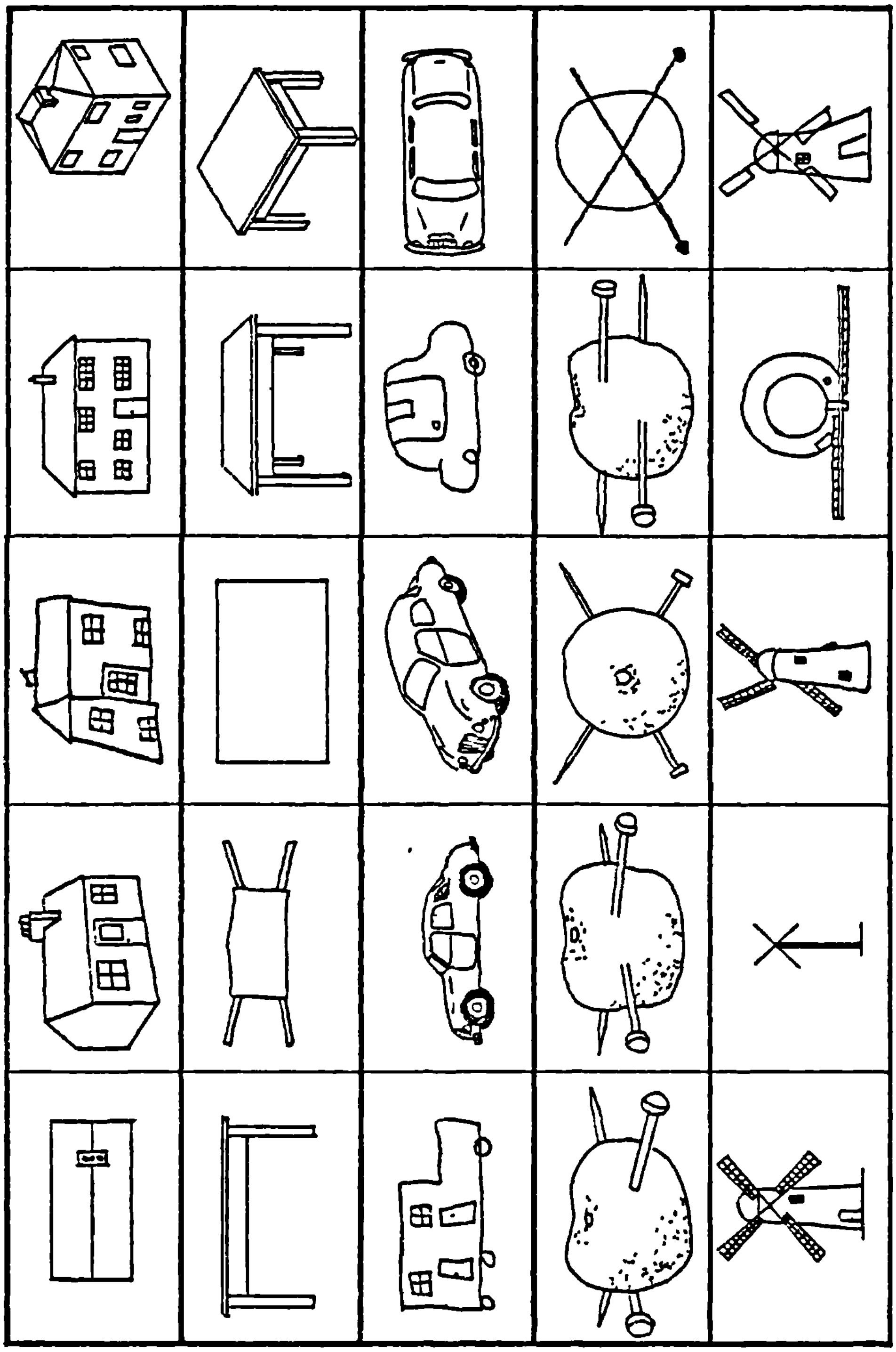
Items 16, 17.

Figure 23. Post-test assignments (Items 16, 17)



Items 20 to 22

Figure 23. Pretest and post-test assignments (Items 20-22)



Items 26, 27.

Figure 23. Pretest and post-test assignments (Items 26, 27)

Table 4

Pretest and Post-test Instructions

Item No.	Instructions
1	Put a cross on the one which looks most like a house.
2	Put a cross on the one which looks most like a table.
3	Put a cross on the one which looks most like a car.
4	Put a cross on the one which looks most like an orange with two knitting needles through it.
5	Put a cross on the one which looks most like a windmill.
6	Draw a straight line in each of the five square boxes: one line in each box. Make all your straight lines different.
7	Draw a straight line in each of the five circles: one line in each box. Make all your straight lines different.
8	Here are some letters and shapes. Use a felt tip or a crayon to colour parts which are straight. Do not colour the curved parts.
9	In the box draw a triangle different from this one. In the next box draw a quadrilateral (a shape with four sides) which is unusual. In the next box draw a quadrilateral (a shape with four sides) which is interesting to draw.

Table 4 continued

Item No.	Instructions
9	Divide the next box into seven regions using three straight lines. Divide the last box into seven regions using two curved lines.
10	One of the drawings in A is of a different sort of object. Put a cross on this drawing. In B, decide which of the drawings is the odd one out. Which one is a drawing of a different object? Put a cross on this drawing. Do the same for C, for D, for E.
11	Here is a drawing made from milk straws. Put a cross on the drawing below which is a drawing on the same model.
12	(i) Here are three dots. Which dot is nearer the middle dot? The dot on the left or the dot on the right? Put a tick by the word <u>left</u> if you think the left one is nearer. Put a tick by the word <u>right</u> if you think the right one is nearer. Put a tick by the word <u>same</u> if you think they are the same distance from the middle dot.
13	(ii) Decide which is longer. Put a tick by <u>a</u> if you think <u>a</u> is longer, a tick by <u>b</u> if you think <u>b</u> is longer, by <u>the same</u> if you think they are the same length.
14	(iii) Instructions as for part (ii).

Table 4 continued

Item No.	Instructions
15	(iv) Here are three distances between dots. You can see the dots at the points of the arrows. Write down what you think about the distances <u>a</u>, <u>b</u>, and <u>c</u>.
16	Here is a drawing of three tall buildings (trees) on a flat area. (i) Decide which is tallest. Put a <u>t</u> on this building (tree).
17	(ii) Decide which is shortest. Put an <u>s</u> on this building (tree).
18	(i) Draw a picture of a cup and saucer as it would look like if you were looking at it from directly above.
19	(ii) Do the same for a television set and a table.
20	Here is a drawing of a large factory. There are rows of windows along the front but they are not drawn in. Draw in about six windows so that the drawing looks right.
21	There is a road in front of the factory going along by the factory and by the trees further along the road. Draw in the road. Put a cross on the middle of the front of the factory.
22	There should be lamp-posts along the edge of the road. Draw in three of these. A helicopter is just behind the factory. Draw the part of the

Table 4 continued

Item No.	Instructions
22	helicopter you can see.
23	Imagine you are the pilot of a Concorde just coming in to land on a runway. Draw a picture
24	of the runway and the airport buildings as they would look like from the aeroplane.
25	Imagine you are a toy soldier on the edge of an empty chessboard. Draw a picture of what the chessboard looks like from there.
26	Put a letter p on the plan (the view from above) of the house, ... of the table, ... of the car, ... of the two knitting needles through an orange ... of the windmill.
27	Put a letter e on the front or side elevation (the view from the front or the side of the house, ... table, ... car, ... orange, ... windmill.

Pretest marking scheme. An attempt was made to produce a satisfactory marking scheme which would be objective for all the items in the pretest. Such a marking scheme has several advantages. It entailed classifying the responses in a structured way enabling children to be compared quantitatively. It also had the merit of making the marking rather less subjective. Each item, except number 24, was scored by an integer from zero to ten inclusive. This allowed comparison between items as well as between children and enabled some statistical data to be accumulated. Objections can be raised about the employment of this procedure. It could be argued that such a grading scheme is arbitrary. The investigator is of the opinion that, in spite of the extra work involved, such an exercise is worthwhile and throws up points of interest which are otherwise likely to be overlooked.

Such quantifying marking schemes seem to imply an underlying continuity in many cases and it would appear that such is the case.

Table 5

Pretest and Post-test Marking Scheme

Item No.	Marking Scheme T = 10 (ten)
1	(C) 2; (A) 3; (D) 5; (E) 9; (B) T.
2	(B) 2; (C) 3; (A) 5; (E) 9; (D) T.
3	(D) 2; (A) 3; (E) 5; (B) 8; (C) T.
4	(E) 2; (B) 3; (C) 5; (B) T; (D) T.
5	(E) 2; (B) 3; (D) 5; (A) 9; (C) T.
6)) 7)	Some curved lines used, 2; Four straight lines with some repetitions, 4; Four straight lines all different, 6; Five straight lines with some repetitions, 8; Five straight lines all different, T.
8	All straight lines correctly shaded, T; For each incorrect response -1 down to zero.
9	For each of the five items 0 incorrect response 1 for partially correct, 2 for correct.
10	Each correct response 2; Each incorrect 0.
11	Octagon, 2; square, 4; pyramid, 6; figure with one missing line, 8; correct,T.
12	Left, 3; right, 7; the same, T.
13	a, 3; b, 7; the same, T.
14	b, 3; a, 7; the same, T.
15	<u>a</u> or <u>c</u> longest, 3; a>b and c>b, 5; b is smallest, 6; a=c and a>b, 7; all equal T.
16	Tallest: right, 3; all equal, 6; middle, 7; left, T.
17	Shortest: left, 4; middle, 5; all equal, 6; right, T.

Table 5 continued

Item No.	Marking Scheme
18	An attempt to draw an elevation, 1; a circle on its own, 2; cup drawn but no saucer, 6; plan but cup has no handle, 7; plan but cup has poor handle, 8; a reasonable plan, 9; a good plan, T.
19	Both television set and table are logical constructs, 1 or 2; both elevations, 3; only one plan, 4; T.V. plan and table as elevation, 5; T.V. plan and table with legs showing, 7; a reasonable plan, 9; a good plan, T.
20	Windows: congruent windows with the top and bottom drawn parallel (depending on the amount of perspective attempted), 3 or 4 or 5; receding lines used (depending on amount of perspective), 8 or 9 or T.
21	Street: curved and not converging, 3; straight with parallel edges, 5; curved and converging, 6; straight slightly converging but not enough, 8; straight converging almost enough, 9; straight converging exactly, T.
22	Lamp-posts: leaning, 2; upright, 4; Distances approximately correct, an additional 3; smallest lamp-post is on the left, another additional 3.
23	Runway: poor attempt, 1 or 2; straight and parallel, 5; roughly converging, 6 or 7 or 8; receding and straight, 9 or T.

Table 5 continued

Item No.	Marking scheme
24	Buildings: sideways on or an elevation laid flat, 1; front elevation correctly oriented, 2; reasonable perspective, 4; good perspective, 5. (For this minor item a maximum of ten seemed inappropriate).
25	No perspective, 1 or 2 or 3; poor perspective, 4; slightly better perspective or front elevation, 5 or 6; reasonably good perspective, 8 or 9; good perspective, T.
26	Each correct response, 2.
27	Each correct response, 2.

End of table 5

Pretest and post-test results. The results of the test in the pilot class, class B, are given in appendix 6. The test was administered to the corresponding control class, class A, but it was decided that little extra information would be gained by analysing the results for this class.

It may be seen that the results of class B on the pretest are worth considering in some detail. Items 1 to 5 scored high means as was intended as were items 6 to 11. It is evident that some eight year old children could cope adequately with these questions and that ten year olds would be likely to do as well.

The rest of the items fell into rather different categories. Item 17 proved ridiculously easy due to a fault in the design of the item. Item 26 was found easy by the pilot class, due partly to plans being dealt with very thoroughly by the class teacher immediately prior to the pretest. It is likely that other eight year old classes would find this more difficult.

This may also be true for items 18 and 19. The rest of the items, namely 12, 13, 14, 15, 16, 20, 21, 22, 23, 24, 25, and 27 give means between 4 and 8 (except for item 24 scored out of 5). These seem to be of a standard difficult enough for eight year olds.

Variations in the responses were highlighted by calculating the standard deviations for each item. Item 16 was scored consistently and incorrectly but greater standard deviations were found for most of the other items.

The pretest showed quite similar results in the two ten year old classes, see appendix 6. . Both the means and standard deviations on items 1 to 11 show very little variation in all three classes, with most scores being quite high suggesting that these tests were well within the capacity of these eight and ten year olds.

For the rest of the items, although there are occasions in which one class scores higher than the others, the marks are remarkably consistent. Sometimes one class scored better, sometimes another with no obvious pattern emerging. It was decided, in the light of these results, to set the post-test on substantially the same lines using only the later items, numbers 18 to 27, and to be concerned only with the ten year old classes S and T, the experimental class S being the only class to be given the curriculum in a form resembling its final version.

In the event, the post-test administration and the teaching of the curriculum materials was complicated by arrangements made within the school between one academic year and the next. The year containing the experimental class S had two parallel classes, the control class T, and another class U. These were divided into four classes by the extraction of some of the more able children from each class. The post-test, therefore was given to those children who were in class S and class T but for comparison purposes only those left when the brighter children were removed were available for testing purposes.

Thus the pretest, post-test evaluation was administered to children of average and below average ability in an inner city school. The results of the post-test on the residue of classes S and T (labelled class L and M respectively) are given in appendix 6 also . The post-test was administered four terms after the pretest.

For each item each child's pretest and post-test scores were taken as a matched pair and a t statistic produced for the items 18 to 27 inclusive. Table 6 lists the items used in the post-test and table 7 gives the results.

Table 6

Items analysed in Pretest, Post-test comparison

Item	Description
18	Freehand drawing of a cup and saucer.
19	Freehand drawing of a table and a television set.
20	Drawing in missing windows on a building.
21	Drawing in a missing street in front of a building.
22	Drawing in missing lamp-posts along a street.
23	Freehand drawing of a airport runway.
24	Freehand drawing of airport buildings (max 5).
25	Freehand drawing of a chessboard.
26	Selecting plans from various drawings.
27	Selecting elevations from various drawings.

Copies of children's work on these activities may be obtained on request from the author.

Table 7

Pretest, post-test t statistics for matched pairs for the experimental and control group

Item	18	19	20	21	22	23	24	25	26	27
Experimental group (n=16)	** 2.9	*** 4.5	*** 4.9	* 1.9	*** 4.9	*** 9.4	* 1.9	0.8	0.2	* 2.6
Control group (n=21)	-0.3	-0.5	1.7	-1.0	*** 3.8	*** 4.4	0.6	1.3	0.4	0.6

* $p < 0.1$
** $p < 0.01$
*** $p < 0.001$

Of the ten items the experimental group scores improved for eight of them significantly ($p < 0.1$). One of which was highly significant ($p < 0.01$) and a further four were very highly significant ($p < 0.001$). The conclusion is that these improvements were unlikely to happen by chance, neither did they occur due to the increased maturity of the children. This was shown by the t values for the control group. The experimental group which had been given explicit and implicit projective activities improved significantly in the accuracy of their structured and unstructured drawings and in their perspective selections on the post-test.

Sometimes an improvement may take place in the absence of a projective curriculum as is evidenced by items 22 and 23, though even here the t scores of the experimental group are appreciably higher than those of the control. Why this should be so in the case of these items is far from obvious, especially as similar exercises did not show the same results. Perhaps some form of transfer of learning was responsible. The results of items 25 and 27 pose questions. The pretest in item 27 was presumably too high to allow for much improvement and the freehand drawing of a chessboard in item 25 appears too difficult. It seems likely that the unnatural posture required to squint along a chessboard ensures that this experience is uncommon and that the convergence of the receding lines is not very obvious and difficult to internalise because of the limited length of the board compared with, say, a straight road or an airport runway where the convergence may be more easily "seen".

It may be concluded that the test instructions were not often misinterpreted, in spite of the need for them to be brief and concise. The children responded in ways which generally vindicated the marking scheme; not only did it prove possible to use the prescribed scheme, but the results were meaningful and capable of sensible statistical manipulation.

The Main Research : Classification of Children's Responses

The main programme. The teaching of a curriculum in projective geometry took place over a period of just over three terms. Some of this initially was in the experimental class, (class S); and then later with a residue of class S, (class L), with the more able children extracted to become part of class P. The opportunity was also taken to use the same material with class P though the ideas were fresh to the majority of the children. The teaching mostly took the form of providing individual work sheets which were then collected for analysis. An aim was to provide a common content whilst enabling individual responses to be made by the children. A few activities involved group work and fewer still, work for particular individuals.

It needs to be stressed that the investigation was throughout at the exploratory stage. Assignment sheets were constructed and reconstructed in the light of the children's responses and as further avenues of research were newly considered starting from the known Erlangen classification the activities were reclassified over a period into the final form specified in the fifth chapter on geometrical thinking modes.

Children's responses were considered thematically in four ways:

(a) Geometrically, making judgements about geometrical structures and geometrical thinking; (b) Psychologically, considering children's conceptual problems, probing along specified dimensions, and classifying understandings; (c) Pedagogically, attempting to identify possible critical phases in the timing of activities, analysing the complexities of diagrams, and considering the need for prerequisite skills; (d) Managerially, adjusting the instructions for class use, modifying the organisation of the scheme, and specifying particular skills required for certain exercies. With hindsight the approach used at the beginning was rather simplistic. A series of disjoint sets of activities on the themes were analysed described as: (a) topological; (b) projective; (c) affine; (d) similar; and (e) congruent. An analysis of these themes and children's responses to the associated activities are given below. The first appendix contains all the activities in their final form.

A topological theme. It was originally envisaged that the usual sort of topological activities would provide a useful background for children, considering such things as order along a continuous curve, being inside or outside a closed two dimensional or three dimensional space, prior to some explicitly projective exercises. Research in the geographical area as outlined in chapter two suggested that a consideration of children's cognitive maps would provide greater opportunities for investigation and interpretation. It was hoped such activities would give greater insights into children's thinking in this area and have at the same time potential links with a theme in similarity through the idea of an elementary map.

The four assignments given in the appendix under the title (A): Perception of the Environment are: (a) My way to school (A1); (b) My home (A2); (c) Where are things? (A3); and (d) What can you see? (A4). These activities are very different from each other. They provide a variety of opportunities for the child to show differing perceptions of the environment and for the teacher to become aware of children's interpretations and understandings.

Children's responses to My way to school were as follows: one child out of 28 used a purely topological approach (1/28); one vaguely projective (1/28); too poor to classify (1/28); a mixture of plans, elevations and iconic symbols (15/28); a mixture of elevations and a topological approach (10/28). To My home the responses were: elevation only (5/29); plan only (6/29); plan and elevation (5/29); a logical construct with all four walls showing or at least two opposite ones (4/29); unclassifiable (1/29); poor perspective (2/29); and reasonable perspective (6/29).

The activity What can you see? was used by the class teacher using a structured writing exercise requiring answers to set questions. This writing was well done by the majority (18/22); reasonably well done (3/22); incomplete (1/22). Only a few children responded to the challenge of drawing a picture from another view but perhaps surprisingly most of these made a reasonable attempt (7/9); an elevation was drawn by one (1/9); and a building climbing into the sky by another (1/9). The pilot class B also tried this exercise using the drawing of a farmyard. They were asked to draw the view from a bedroom in the farmhouse. They were also asked to write about the view. They responded as follows: plan with icons (14/37); slight perspective (11/37); better perspective (6/37); unclassified (6/37). All the children were able to catalogue objects which could be seen from the window. When asked to draw a plan of the farmyard class B produced: a well co-ordinated plan (6/29); a poorly co-ordinated plan (9/29); some aspects of elevations (6/29); and a poor attempt (8/29).

Geometrical analysis suggested that the children used whichever type of geometrical transformation seemed to be most appropriate for the purpose in hand. Visual "snapshots" involving topological or perspective ideas, plans and elevations are used as seem most fitting.

The assignments proved very variable in difficulty. The intention was to yield information about their thinking, but the conceptual challenge proved too great for many of the children. Conceptually the logical problem of transporting themselves in their imagination within a perspective view and to visualise looking out was too great. It may be possible to structure the exercise by giving a window frame, curtains and handles and ask for a drawing of the view through the glass. In this respect the Piagetian view of co-ordination of views seems correct in spite of Donaldson's misgivings. It should be noted, however, that the children appear quite resilient - perhaps they are used to impossible demands - very few gave up though many of them expressed or showed dissatisfaction with their attempts.

Pedagogically few changes were made. The general plan enables children to "show what they can do and how they are thinking" and this remains as the main purpose of the exercise. The pictures are a resource for the teacher which may well be used for other, possibly more suitable, purposes. It should be noted in passing, that although the tasks might be assumed by the children or the teacher to be "busy" work, the children greatly enjoyed and took trouble in colouring in the perspective view of the bungalow, for example, and this may be a useful exercise to be used to familiarise themselves with the essential features of the drawing, prior to the introduction of the main exercise.

Managerially, a few adjustments were made to organise the worksheets. The headings: aim; process; materials; what to do; motivation; learning; display; discussion; vocabulary; enrichment; minimum expected of children; evaluation; were finalised here by suitably modifying Larkin (1981) though not all assignment sheets used all the headings.

A projective theme. For some time it has seemed to the investigator that a euclidean approach to straight lines is conceptually complicated.

This theme was developed at length in chapter two. The projective approach to straightness makes more sense in terms of geometrical structures with projective geometries being the most general in which straightness is a concept, uncluttered by notions of parallels, angles and lengths.

The assignments here are closest to a structured sequence of learning activities where succeeding items build on to the previous ones. For this reason the activities start with very easy ideas, develop through notions of flatness and paper folding to the construction of polygons and the outlining of competitive two person games. Further it was in the nature of the exercise that children should take part in discussions, experiment with torn newspaper, and play games which are difficult to record. The topics covered were basically: (a) Straightness; (b) Flatness; (c) Regions; (d) Triangles and other polygons; (e) Constructions using Pappus' Theorem; and (f) Games using Desargues' Theorem. Straightness did not cause many problems for the eight year olds. The discussion of flatness and other meanings of the word "flat" was lively and sensible. The ruler skills required for Pappus' Theorem were found by many to be too difficult, but they were somewhat intrigued by the result especially when the six original points lie on a circle.

These exercises are reproduced in the first appendix under the title (B): Straightness and Polygons. Which path? (B1) was tried by the investigator with two individuals. It was found to be rather trivial for these less able eight year olds but was retained for two reasons; it is useful for a teacher to know that children can do this exercise, and the exercises at the end tend to highlight small problems the children may have when a line is partly straight and partly curved. This problem was also the subject of Which parts are straight? (B3) (item 8 on the pretest exercise) where a mean of 7.2 with the pilot class of eight year olds implied some children may have problems. There is also a link with Straight (B2) (Pretest item 7). Geometrically children should be encouraged to see a straight line as "the shortest distance between two points" and as "the line with constant direction", although neither of these is a projective approach to the concept.

The activities Joining points (B5) and Joining more points (B7) use a projective approach and caused some problems. Children found it difficult to fold a piece of paper so that both marked points were on the fold. It seemed useful to persevere a little and give children a chance to perform this skill adequately. Children here should also be encouraged to see a straight line as "the intersection of two planes", as "the line given by a stretched string", and as "a line of sight".

Pedagogically and geometrically the consideration of straightness and flatness progresses to regions - one fold leads on to two and then onto polygons when three or more folds are used. The freedom to experiment and throw away failures literally into the waste paper basket was an unusual aspect of these activities. Regions (B8), Triangles (B9), Triangles and quadrilaterals (B10), More triangles, quadrilaterals (B11) (Pretest item 9), More regions (B12) and Yet more regions (B13) were used with class B.

Originally too many activities were allocated to individual assignments and these were broken up into smaller, more manageable, items for easier use in the classroom. The children particularly enjoyed constructing and colouring regions formed by folding lines and these made an attractive display. Later Constructing quadrilaterals (B14) and Drawing straight lines (B15) were added but not tested in the classroom as these are normal items in a mathematics curriculum, a difference here being the projective approach to the activities rather than a more usual euclidean, though it is unlikely that children would be aware of this distinction.

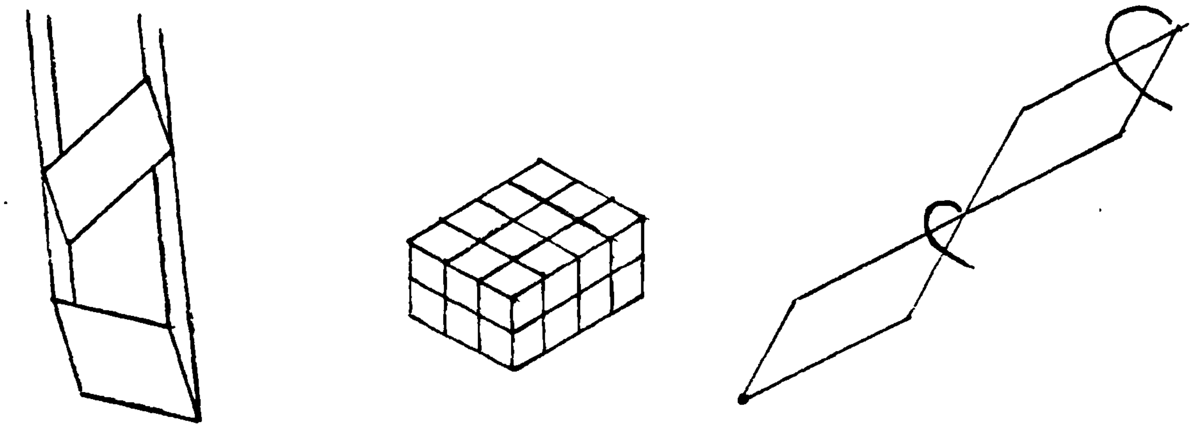
Pappus theorem was used in New lines from old (B16) and More new lines from old (B17) to enable children to practice ruler skills. With some emphasis on the care needed to ensure the constructed lines are in the correct place, it is possible for eight year olds to get the new straight line joining points 7, 8, and 9. The result is rather more unexpected when the six points initially lie on a circle; one of the main problems here is to ensure that children have a good circle at the start. A handout of appropriately drawn and marked circles is better than children constructing their own by drawing round the edge of a circular object.

An approach through games was used in Special noughts and crosses (B18) and Special line noughts and crosses (B19). These have been described elsewhere in Salisbury (1982) and a copy included in the appendix 7.

Although the exercises on straightness are generally brief, a large number of them tend, unfortunately, to emphasise a two dimensional approach. Other activities in projective viewing were designed to counterbalance this emphasis. They involve considerations of two dimensional representations of three dimensional objects. A set of them which are in the first appendix under the title Projective Viewing (C) involve selection processes, namely: Which is the same? (C1) (Pretest item 11), Which are drawings of the same? (C2), Odd ones out (C3) (Pretest item 10), More odd ones out (C4), Which is the tallest building? (C10) (Pretest items 16 and 17), and Which is the tallest? (C11) (Post-test items 16 and 17). The previous section of this chapter contains an analysis of the pretest items which generally were well done.

Others were produced on a similar theme requiring group or individual work. These were not given to the experimental group as it was considered better to give to a group of selected children and deal with them separately. An account of this is given in the next section of this chapter. They are mainly environmental in approach namely: Shadows from a lamp (C5), More and yet more shadows from a lamp (C6 and C7), Skeleton drawings (C8), Drawing through a window (C9), and Viewing through a paper frame (C12).

An affine theme. After some consideration specifically affine activities were rejected. Affine transformations preserve parallelism and ratios in directions. Specifically affine activities could have included such experiences as using shadows from the sun (involving parallel projections), using and constructing isometric drawings, and constructing parallelogramic grids (such as used in a pantograph).



Sun shadow of a parallelogram — Isometric drawing of a 4 x 3 x 2 cuboid — Enlargement by a pantograph

Figure 24 Three types of affine transformation

The sun shadow activities were rejected after the experience gained in the preference experiment in chapter three. As there appeared to be no strong preference for a projective compared with an affine transformation it was decided that they would tend to duplicate the previous projective shadow activities. The isometric drawings fall into a different category as measurements of length are needed as the name implies. Giles (1979) has already dealt with these thoroughly. These activities seem more appropriate for older children. Affine views are considered, however, in the co-ordination of views theme especially in Making more drawings look right (F6) discussed later. The use of a pantograph was omitted because it is difficult to use effectively especially with apparatus made by children which lacks sufficient rigidity. It also merely gives a similar transformation, an enlargement with linear magnification of two 2 (or one half), and the projective implications are therefore minimal.

When affine transformations were being considered, the original scheme was marginally reorganised to incorporate similarity and congruence activities within the set of affine exercises. The investigator became of the opinion that such an inclusion was necessary and would help to prevent the proposed scheme from becoming unwieldy. Most, if not all, of the usual exercises in shape and size could be omitted. To incorporate these into a scheme would not only be unnecessary but would disguise and detract from the projective nature of the main activities.

Soon afterwards it became apparent that a reorganisation and restructuring of all the themes was certainly desirable and probably essential.

The themes were then reorganised and activities redistributed in some cases to fit the new categories which were: Topological perception of the environment; straightness and polygons; projective viewing; and affine representation.

By this time also two new themes had been added to the scheme namely: visual illusions and co-ordination of views. These are discussed later. They were seen as additions originally but later were absorbed into the scheme as an integrated part of a more valuable and wider reaching curriculum.

The activities in affine representation were: Which are views from above: (D1); Which drawing looks most like? (D2) (Pretest items 1 to 5); Views from above (plans) (D3); Drawing plans (D4) (Pretest items 18 and 19); Which is the plan? (D5) (Pretest item 26); Front views (D6); Which is the front view? (D7) (Pretest item 27); Solids from drawings (D8); Models from drawings and plans (D9); Using more plans (D10); and Which drawing looks most real? (D11).

Some of these items were developed late in the investigation, often in response to children's reactions to other exercises previously used in teaching sessions and in the pretest. These later activities were tested after the main stage of the research and are those numbered (D1), (D3), (D6), (D8), (D10) and (D11).

The activity Models from drawings and plans was tried out on the pilot class, using the plan and line drawing of a village. The intention was to make the exercise a class activity; some children making a windmill, some schools, some churches, and some a base upon which to set the other models. It soon became evident that such activities require the continued presence of a class teacher and are inappropriate for visits of an hour a week. Other problems were encountered. The lack of skill in using scissors and glue in model making was apparent. The model making involved considerations of dimensions and relationships between parts of the buildings. The children enjoyed this aspect. The problem of scale also arose when large churches dominated bases on a smaller scale! After three visits it became evident that the piecemeal and isolated nature of the activities was causing problems and the final model never came to fruition. Nevertheless it was retained as resource material. Using building blocks for the main buildings rather than, or as well as, paper models, emphasising the need for compatibility of scales, could turn this into a successful venture. It would become a more suitable exercise when it is an integral part of a broadly based curriculum topic such as might arise from a school visit to a place of interest especially when the available literature may contain both aerial views and plans. This has the added advantage that the children will have been able to explore the area under consideration.

In retrospect, the affine approach to projective ideas was, to the investigator, the least satisfying. The decision to lead into similarity and congruence through parallel preserving transformations is not easy to achieve, the simplistic Erlangen programme approabh has severe limitations in this area. The differences between affine representations, especially isometric, and projective appear to be rather elusive for many primary children. The emphasis in this section is on plans and elevations and although I believe of value, it did tend to distract from a satisfying overview. The relationship between this approach and the normally used euclidean design needs further investigation. It is hoped that the modified scheme in affine ideas contains activities leading to model making and map making which will involve the children in similarity transformational aspects and will then lead naturally into more usual aspects of primary geometry.

A theme in visual illusions. The relationships between visual illusions, perception, perspective, and projective geometry were discussed in chapter two. Visual illusions were chosen which depend on extraneous lines often suggesting impressions of depth. For these a discussion and an examination of the problems and possible solutions are suggested. Other illusions were chosen because they use an affine rather than a projective representation, that is, lines which are actually parallel in the three dimensional solid are drawn parallel in the two dimensional representation rather than converging as the lines recede further from the eye. To investigate this further projective as well as affine representations were used. For example, two versions are given of a Necker cube, one retaining parallels, the other having the parallels drawn as receding lines.

From illusions it is a natural step to develop experiences in perspective. Structured drawing is the medium used to consider inserting extra details in some diagrams. Complete receding lines, parts of some receding lines, and intersecting diagonals are required. The assignments are: Seeing things (E1) (Pretest items 12 to 15), Looking at things (E2), Looking at more things (E3), Lines converging at a point on the horizon (E4), Inserting missing objects (E5) (Pretest items 20 to 22), Where is the middle? (E6). Apart from the items used in the pretest the assignments here are not easily analysed because of the individuality of the responses. Looking at things was taken in conjunction with Looking at more things. The first of these was introduced by exposition and discussion and written comments from children were not requested. The main intention was to stimulate delight, discussion, and some confusion. Looking at more things was then given and the children asked to respond to specified questions in writing. Points worth noting are: On diagram 2 most children did not mention the distortion evident when the diagram is rotated (13/18); the rest (5/18) mention problems: "Upside down looks the same as the right way round. Apart from being more wonky". In diagrams 3 and 4 most children say nothing wrong, (one interpreted these as a series of shapes in two dimensions and not as a three dimensional object, and was thus correct in not seeing anything wrong!) (16/18), though some had a few reservations: "they look as though they are turning round" and "they spiral in a triangle". The other two (2/18) were worried by the diagrams. "When I draw things I can never see all the edges". Diagram 5 produced much the same reactions with only a few children complaining about climbing stairs and not getting anywhere. "... looks alright but it is a strange kind of puzzle". The children appear to focus on the localised compatibility rather than the overall incongruities of a diagram.

Diagram 6 was usually taken to be a corridor with doors. Diagrams 7 and 8 are late additions and were not tested. Lines converging at a point on the horizon (E4) showed a number of variations by the responding children. About a third correctly extended the lines to meet at the vanishing point (16/47). A further third made them meet about halfway between the vanishing point and the boundary of the drawing (17/47). Some had the lines meeting on the boundary (5/47), one beyond the boundary (1/47), one kept the lines parallel (1/47), another diverging (1/47), two took the road round a corner (2/47), one had the lines converging below the vanishing point (1/47) and the remainder did not respond (4/47). Some children were unhappy with their responses realising that their extensions made the road appear to go "uphill". For this reason an extra diagram was produced in the final version to provide further practice.

In Where is the middle? one third of the children chose the correct response. The exercise was modified to ask for a free response, to also give experience in finding the middle of a rectangle by drawing in the diagonals. "Are the goal posts the same size?" brought similar responses to those found previously. "Yes, because the far goal has to be smaller because the pitch goes smaller" (sic) and "no because I measured them" were typical responses.

A theme on co-ordiaation of views. Of the themes chosen, this is the last and most problematical. In some senses it follows from the previous work and may be considered a natural extension of activities in perceiving the environment topologically considering properties of straight lines, drawing perspective views, considering visual illusions and affine representations. As such, it may be considered, as Piaget and Inhelder (1956) suggest, to be the climax and the final stage of the conception of space. On the other hand such a co-ordination of views may well tax many adult's powers of visualisation, and any exercises produced for children may easily prove too difficult. There is the danger that inappropriate challenges could be made which would adversely affect the children's preparedness to venture into visualising things for themselves. There is a clash of interest here which is not easily resolved. As has been mentioned previously Cox (1977) and Donaldson (1978) also suggest that the problems are more complex than might be imagined.

It was decided by the investigator to: (a) Keep this section short to prevent unsuccessful children encountering cumulative failures, (b) present opportunities for drawing which would both revise previous activities and give the teacher indications of individual children's thinking processes in visualisation, and (c) would present meaningful challenges in ways which demand analyses of situations.

The exercises were labelled: Where are they? (F1), Making a drawing look right (F2) (Pretest item 23), Tiled floors and chess-boards (F3), More chessboards (F4) (Pretest item 25), More drawings and paintings (F5), Where are you? (F6), Which drawing is from which place? (F7) Making more drawings look right (F8), Drawings of large cubes (F9), and What will it look like from other places? (F10).

Where are they? (1) follows on from Where is the middle? in the previous section providing a link between the two themes. Allowing reasonable margin of error - about one eighth of the length or width of the field both the middles, middle of the field and the middle of the left hand near quarter were selected by 8 out of 48. Only two children got both right and only one child drew in the diagonals of the field. This compares with one third choosing the correct place when selecting one out of five possibilities. Exercise (3) was added later to build in more experiences of looking at and drawing parallel lines when receding. As is clear from exercise (4) in both the pretest and post-test, this exercise had difficulties for children and the need for further structuring is evident. A further exercise (number 5) in plans and elevations considered from a co-ordinated view-point is used to revise and extend previous work in the area. Similarly Where are you? extends the use of a sketch and plan previously used in model making. Here the emphasis is on co-ordination. A similar state of affairs occurs here as on previous occasions. So many clues are available for the positioning of the cross. There may be a verbal confusion here as some children put the cross on top of the church (perhaps the correct place for it!) The variety in the responses make any analysis of doubtful value. Clues which may be used are the position of the road (and the T junction), the "diagonal" rather than a sideways view of the church, the absence of such highlights as the windmill and the school, the confusion of the representation of the church tower as a square on the plan, the incomplete nature of the sketch, and the placing of both diagrams on one sheet of paper preventing the rotation of one whilst the other sketch is unmoved. Somewhat arbitrarily the plan was divided into nine regions by trisecting both the length and breadth. The regions were labelled A to I as on figure 25. Of the 29 children only two choose a place in A the correct choice. Most popular were D (8/29), G (9/29), and E (5/29). B, C, and H shared 5/29 with no-one selecting F or H. From these results it may be

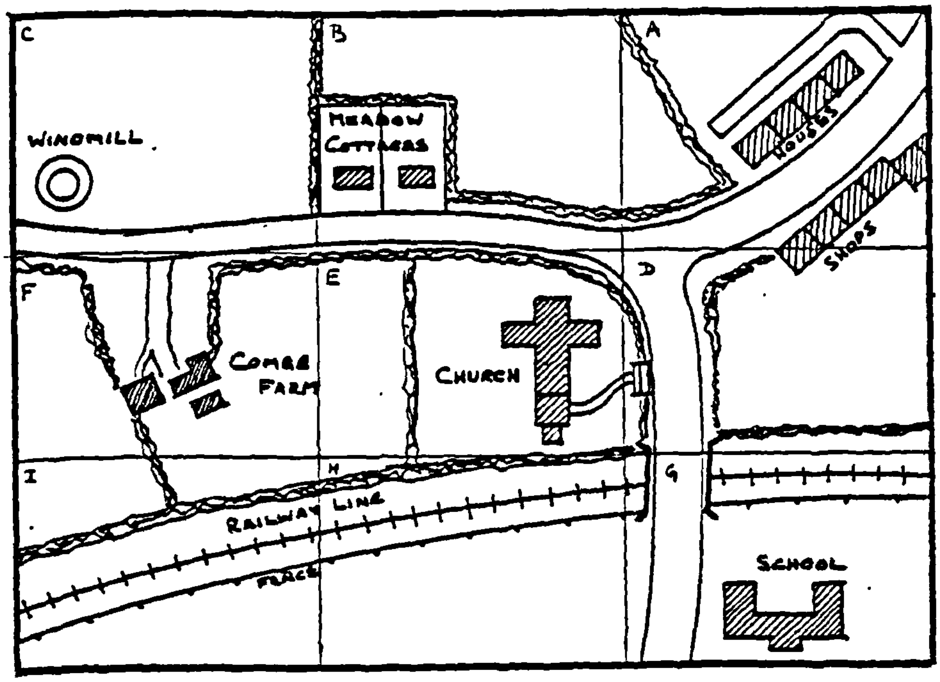
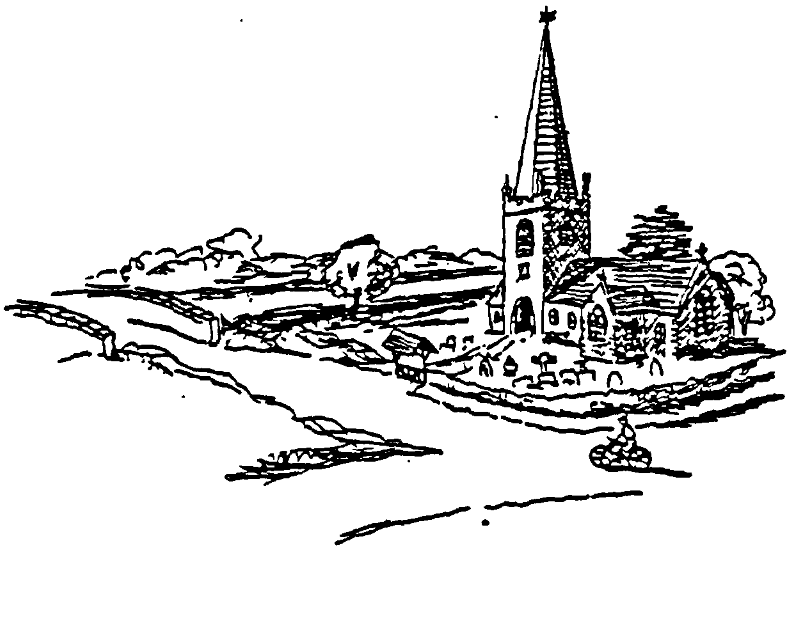


Figure 25. Co-ordination of views. Where are you? Nine regions on the map of the village.



concluded that children use the hint of the road running in front of the church but cannot manage to change their viewpoint to the top of the plan at A. Subsequently two less able children were interviewed individually. Before the question was posed salient points both on the plan and on the sketch were pointed out. After questioning on the lines suggested in the final version both were able to specify A as the correct region. Later this same exercise was given to various aged children in the I.L.E.A. The results of these are given in table 8.

Table 8

Results of 144 children on Co-ordination of Views

Activity <u>Where are you?</u>

Age	A	B	C	D	E	F	G	H	I	TOTAL
5	-	-	-	1	1	-	1	-	-	3
6	-	1	1	3	4	-	1	1	-	11
7	1	-	-	3	1	-	-	2	-	7
8	-	-	-	-	-	-	2	-	-	2
9	6	-	-	5	9	2	5	3	-	30
10	8	5	-	9	11	3	7	5	4	52
11	4	2	1	16	11	2	1	1	1	39
TOTAL	19	8	2	37	37	7	17	12	5	144

A different approach was taken in <u>Making more drawings look right</u>. Originally the children were presented with isometric views of collections of cubes and asked to draw projective views of the solids "which looked right". They were given the opportunity to use a connected dotty projective grid. As the investigator was informed by one child in no uncertain terms "this is to hand"! (sic). A good attempt was obtained by a few (7/53), a reasonable attempt by 3 (3/53). A three dimensional drawing of some sort was obtained by

12 using the grid (12/53) but some only obtained a two dimensional representation (3/53). Ignoring the grid, a few obtained a diagram with three dimensional connotations (8/53) and with flat aspect (7/53). Of the remaining eleven some were unclassifiable, being mostly incomplete (4/53) and the rest (7/53) made no attempt. These results highlighted the need for a better structuring of the exercise with extra visual hints being given as to what "looking right" might mean in this situation. To enable children to come to grips with this more easily the first activity was essentially two dimensional using the idea of square fields which was researched in a pilot task. In the following three dimensional activity Drawings of large cubes, only two cubes were used as parts of a building. The cubes were drawn affinely on a perspective view and children asked to draw the two cube building more correctly on a perspective view with the building missing. This activity is analysed later in this chapter.

Testing materials developed late in the research

A new pilot class. As noted elesewhere some of the curriculum materials were developed late in the research. Reflections by the investigator on children's response in earlier test situations led to various modifications. Some were slight textual changes whilst others were more drastic involving reordering and reclassification. Some extra curriculum materials were added to give children further and deeper experiences in the same topics. The slight textual changes did not merit retesting but the additional materials required testing in school. It was not possible to try this new material on children previously engaged in the research, mainly because they were by now a year older and in reorganised classes. A further group of children of a suitable age was chosen as a new pilot class. There were 29 children in this class aged 9 or over. There was an advantage in the fact that these children came fresh to the research and that this was likely to provide more information than would have been the case with older children aged 11 plus to whom the material would be similar to that which they had already encountered.

The additional test material was administered by the investigator and was organised on a similar basis to the original pilot testing. The children had previously met plans and elevations in their normal school curriculum.

The exercises thus tested are given in appendix 1. They were:- Which are drawings of the same? (C2); More odd ones out (C4); Drawing through a window (C9); Which are views from above? (D1); Views from above (plans) (D3); Using more plans (D10); Which drawing looks most real? (D11); Where is the middle? (E6); Where are they? (F1); Which drawing is from which place? (F7); Drawing of large cubes (F9); and What will it look like from other places? (F10).

An analysis of children's responses. The activity Which are drawings of the same? (C2) caused few difficulties. Over half of the children obtained the correct answers (16/29); a few of them mixed up pentagonal and hexagonal prisms (9/29); with the others making more errors (4/29). This activity appears to be pitched at about the correct level of difficulty though, as an afterthought, the addition of several more figures on the same page would have given the exercise more bite for some children.

Selecting More odd ones out (C4) also caused few problems. Four or five correct responses were scored by over half the children (15/29) although the garden roller question caught out the majority. Tests and puzzles of this sort were generally enjoyed by the children who perceived them as interesting challenges. Drawing through a window (C9) was tried on an individual basis by six children. Their drawings showed the usual responses. Whether an activity is structured in this way or is a freer less structured one appears to make little difference. For example, the parallel edges of a netball pitch which the children appear to realise are parallel are drawn as such even though they should be represented by receding lines. The exercise does, however, seem to be useful in providing additional and different experiences.

Only two children found Which are views from above? (D1) difficult. There was no consistency in the incorrect responses. Views from above (plans) (D3) gave similar results. The value of such exercises is that the teacher can identify the few children who have problems and hence they have opportunities to discuss the children's problems with them.

No problems were apparent in Using more plans (D10). It was observed that less than a half used multiplication to obtain the number of chairs (11/27); the others merely counted (16/27). Of greater interest were the responses to Which drawing looks most real? (D11). Only eight chose the perspective drawing (8/27); the affine view, base horizontal and one side "face on", was more popular (14/27). Children seem to perceive buildings face on, simplifying what they observe and transfer this to a selection process as in this activity.

Further evidence for the need for considering projective ideas was given by children's responses to Where is the middle? (E6). Two marks were awarded for an accurate response and one for a near miss. The results are presented in table 9 and show a better performance on the projective than the euclidean transformation; that is at finding the middle of the projective view of a rectangular field rather than an untransformed rectangle. The result supports the view that projective ideas precede euclidean.

Table 9

Scores for Where is the middle? activity

for 29 children aged nine

Score	Frequency	
	Field	Rectangle
2	24	12
1	5	7
0	0	10
Mean	1.83	1.06

The problems that children have in interpreting lines drawings is shown however by their selections for the centre point of a football field in perspective. The choices were: the correct answer C (9/27); D (17/27); and E (1/27). The position marked D was chosen by several for reasons like, (sic) "because I mesured it", "I picked D because I thorgth it was the middle".

Going from a plan to a projective view proved more difficult in Where are they? (F1). Here the responses were: both correct (3/27); pond only correct (10/27); swing only correct (5/27); and both incorrect (9/27). Selecting elevations in Which drawing is from which place? (F7) was well done. Those obtaining correct responses were: 1, (27/28); 2, (25/28); 3, (28/28); and 4, (28/28). More problems arose in chosing a correct plan from three choices. The correct plan was wrongly orientated which makes this quite difficult. The selections were: the correct plan, number 5, (15/28); 6, (6/28); 7, (4/28); and, no response, (3/28). This activity should lead to a lively discussion and then on to children making up their own simpler examples.

Further interesting results were obtained in Drawings of large cubes (F9). The diagram was improved by the majority (19/29); equal numbers gave little difference (5/29); or made it worse (5/29). Most children merely made the roof line more nearly horizontal. Many children drew pitched roofs. This exercise could also lead to valuable discussion.

The final additional exercise What will it look like from other places? (F10) has nine choices for seven perspective views. It is probable that many adults would find this difficult. A mean of 3.6 correct responses (over 50%) is rather unexpectedly high. The results are presented in table 10.

Table 10

Scores for What will it look like from other places?

for 27 children aged nine

number of correct responses	0	1	2	3	4	5	6	7
frequency	4	3	4	3	6	3	1	3

mean = 3.6 standard deviation = 2.8

The exercise is a projective variant of the three mountains experiment

(Piaget and Inhelder, 1956). It was intended to provide a substantial challenge for those children for whom some of the other exercises may have appeared somewhat trivial. It appears that the projective co-ordination of views may be easier for children to handle than the euclidean co-ordination required in the Piagetian experiment. It becomes apparent from this research that further investigation into children's spatial-geometrical thought processes is required. This forms the subject of the next chapter.

Research Results and Interpretations

Operating principles. The main principle used in organising and operating this research into a projective geometrical curriculum was the provision and testing of productive learning activities. It was decided that a priori children should be involved in activities requiring thought and (hopefully) learning. The learning activities provided a vehicle by which the acquisition of geometrical concepts by children could be studied. The danger of forming unnecessary and unwarranted assumptions about such acquisitions needed to be avoided.

An analysis of the pre-requisite skills which would be needed by children so that they might successfully master tasks was considered. Attention was given to the sequencing of geometrical experiences so as to provide cumulative and meaningful learning. The end product was to be a curriculum package. This consisted of a description of a series of classified and sequenced activities. Suggestions for its use were provided. These were tried out in the classroom and as a result of children's responses they were later modified. The alterations included a reclassification, more detailed sequencing, rewording children's instructions, and amending teachers' guidelines.

Early in the study a pedagogic principle was established that the receptive learning of projective geometrical facts would be avoided.

The chief method used was that of probing questioning. It was believed necessary to avoid inhibiting children by suggesting that "right" answers were required of them. Instead a more relaxed atmosphere was fostered where children could "have a go" at questions which challenged them. Children's thinking processes were to be developed through classroom discussion which emphasises probing and challenging children's ideas and not towards reaching teacher directed conclusions.

Relevance was also an important criterion. Projective links were made with other subject areas in the curriculum whenever they appeared useful. For example, plans and elevations were investigated but without the notion of scale and visual illusions were used to challenge children's perceptions. An assignment card format was considered to be most appropriate.

Post-test discussion. It is the investigator's belief that the processes of questioning and challenging children's geometric impressions in the study did actually succeed. It should be remembered that about half hour weekly sessions over less than four terms was the sum total of the experimental group's projective curriculum, between twenty and twenty five hours contact time. Appendix 6 contains the raw data and table 7 the statistical analysis of the pretest, post-test results. It may be concluded that children (in the lower half of the ability range) generally did acquire greater conceptual comprehension in matters geometrical.

Of course the summative evaluation had its limitations. These are discussed in an earlier section of this chapter. The chief objection to the evaluation may be that only easily testable parts of the curriculum were considered. The more tenuous and perhaps more worthwhile aspects remain outside the domain of an evaluation through a test procedure. Another limiting factor was that only activities developed early in the research could be used in the pretest post-test scrutiny. Nevertheless the test covered a wide spectrum within the curriculum and it would be surprising if the results did not replicate under similar conditions.

Assessment of the activities is not easy. Marking schemes tend to be subjective, or costly in time, or perhaps both. Much of the work can be sufficiently assessed by considering what thinking modes children were using and how far they were being successful. This should be sufficient for most purposes. Recording children's achievements is probably best carried out by the collection and retention of various pieces of typical illustrative material. This may be transferred, along with the children, to their new teacher.

General implications. Geometrically, projective ideas provide a link between topology and affine geometry, but it was found that the structure of geometrical hierarchies was not of primary importance in children's learning. The original Erlangen approach was first modified and then discarded. The focus was directed away from purely geometrical structures towards children's ways of geometrical thinking. Presented with a certain problem, question, or challenge individual children react idiosyncratically. Their interpretations, preferences, and responses will depend on the nature of the challenge, its method of presentation, and the children's past relevant experiences.

The transformational aspects of geometry need attention. Such transformations may be from an object to a representation of it, (either topological, projective, affine, or euclidean). It may also be a transformation from one representation to another. Projective geometry also provides links which euclidean geometry does not.

This has important pedagogic implications. The proposed curriculum could be seen by children as something more than exercises on paper.

For example, assignments involving views of villages, bungalows, and plans of buildings appeared to be more interesting than purely geometrical activities about the properties of lines and angles. Geometry is thus linked with the world within and outside the classroom.

The challenges involved in these activities produced an air of involvement and it is the investiggtor's belief that many children thought deeply when presented with this type of work. On occasions children need "visual thinking time", that is, they need to familiarise themselves with a map, picture, or photograph. This may take the form of class discussion which should not be teacher directed towards the establishment of "correct" conclusions. Rather it should be inquiry based: "What can you see?" "Why do you think that?" Shading or colouring pictures may also be the means of providing thinking time.

If the situation is judged rightly the structured activities enable nearly all the children to make relevant responses. Some may be incorrect. Learning seems best to take place when a variety of media is used, drawing, painting, selecting preferences, making individual interpretations, using display, and discussing. Some teachers may well be surprised by what individual children may accomplish. When responding to challenges children can reveal unexpected facets of their thinking. Different approaches may give greater chances to children in finding success. The handling of competitive and non competitive games and puzzles caused few difficulties. The only serious one that of individuals becoming habitual losers. This was avoided by a judicious change of opponents.

Copies of children's work on these activities may be obtained on request from the author.

CHAPTER FIVE

The Promotion of Geometrical Thinking in Primary Schools

An Examination of Problems in Primary School Geometry

Premable. This study attempts to probe the teaching of primary school geometry and to offer some solutions. The principle problems may be defined as follows:

(a) the unique and distinctive nature of geometry

(b) the function of geometry in the school curriculum

(c) the links between geometry and other subject areas

(d) difficulties associated with children's learning of geometrical concepts in relation to the sometimes contradictory findings of research

(e) children's geometric interpretations, their projective preferences and abilities, and

(f) research brought about by the development of a new approach, that is, through a projective geometry curriculum.

As a result of these studies it is concluded that a projective element in the primary curriculum can be both useful and successful. However further research in this area is necessary. It is possible that computer simulation may develop exciting learning possibilities.

The unique and distinctive nature of geometry. The study considers various geometries and their relationships. It suggests that projective geometry can make the link between topology on one hand and affine geometry on the other. An emphasis on affine geometry and particularly its euclidean subgroup is deplored. The affine linear transformations (usually represented by matrices and vectors) or the euclidean subgroup of transformations tend to disguise geometry's spatial nature. They both isolate topology and moreover make geometry algebraic.

Projective geometry can be and should be the link which holds geometry together. Geometry can then be presented with a greater conceptual clarity. It is part of our culture, if it is ignored or denigrated now, our descendants will be the poorer. Geometry is considered as a set of logical structures, as a study of relationships, and as a process in the study of spatial modes of thinking.

The function of geometry in the school curriculum. The broadening of the primary curriculum to include geometry has generally been considered beneficial. Bruner has observed that any subject could be successfully taught at any age providing that the encounters were at a level which could be understood. The present study attempts to do this for projective geometry.

At the present time it appears to the investigator that geometry tends to consist of isolated and unconnected "snapshots". These snippets of geometrical information appear to be of marginal interest outside the classroom and often unconnected with other subject areas. An infusion of some projective geometrical experiences into the primary curriculum should remedy this and make geometry more relevant.

The links between geometry and other subject areas. It is suggested here that links may be made across the curriculum by a judicious use of projective ideas. These links can be a vital aid in providing relevance for general geometrical experiences. Projective geometry has particularly applicable aspects in science, in art, and especially in geography. Such applications of projective ideas enhance both geometry and other areas of the curriculum in a unique way which aids children's understandings. They assist the development of children's visualisations and their spatial perceptions.

Difficulties associated with children's learning of geometrical concepts in relation to the sometimes contradictory findings of research. Problems of children's perceptions of geometrical concepts and associated learning difficulties are reviewed from a Piagetian standpoint. An analysis of his projective research is included under the headings of (a) general spatial abilities; (b) pictorial space; (c) spontaneous and copying exercises; and (d) perspective ability. It is shown that Piaget and Inhelder worked within an Erlangen paradigm but reverted on occasions to a euclidean standpoint. Thus the great contribution which they have made to understanding children's spatial conceptual abilities requires careful consideration in some aspects. Research contributions both supporting Piaget's views and questioning them are considered. Perception has been researched in depth from many different viewpoints and some issues relevant to geometry in general and projective geometry in particular are also analysed in relation to spatial abilities. One of the main problems in reviewing the literature is that few researchers had operated outside a blinkering euclidean paradigm. This militates against accepting their findings. Problems also exist because few people have any experience of using projective ideas in primary schools.

Children's geometric interpretations, their projective preferences and abilities. The research begins with the study of conceptualisation of straight lines and their distinctive projective features. This was used as a fairly minor activity to gain research experience. A pilot class in a city primary school was used to research children's interpretations of line drawings. Several difficulties were experienced and reflections on these led to further delving into relevant research literature. A new direction was then taken with the pilot class in investigating children's perspective preferences. It is the investigator's conclusion that children have (a) a strong preference for projective rather than euclidean transformations and (b) little preference for projective rather than general affine transformations.

These results and reflections on the children's interpretations modified the investigator's approach to the development of the projective curriculum.

Research brought about by the development of a new approach, that is, through a projective geometry curriculum. Originally the scheme was divided into sections having themes in (a) topology, (b) projective geometry, (c) affine geometry, (d) similarity geometry and (e) euclidean geometry. It became evident that such a partitioning of geometry did not fit research findings and that any partitioning had its deficiencies (as other researchers have found). Later two themes (f) visual illusions and (g) co-ordination of views were added and the euclidean theme withdrawn. In addition the projective theme was subdivided into two dimensional consideration of straightness and related ideas, and a three dimensional study of projective views.

Reflections on the research as it progressed led to a further classification on ways of thinking in geometry. This outline is intended to show the formative nature of the research. Children's spatial abilities are not easily studied. It can be asserted that the teaching by the investigator did lead to significant improvements in a pretest, post-test evaluation.

Problems in the Promotion of Geometrical Thinking

Modes of thought. The original Erlangen approach was considered unsuitable, and was made more so by later additions, deletions, and modifications. An alternative structure was required to provide a clarifying overview without which it would be difficult for a teacher to operate. Any structure has its limitations and difficulties, as other researchers also found.

The ways children respond to a task can be very different. They may interpret a task for example, either topologically, perspectively, affinely, or a mixture of these. These modes of thought were used as the structure upon which to hang geometrical experiences. But the confusion remains. It becomes noticeable that the themes as outlined appear not to have distinctly identifiable boundaries. Often there is a gentle transition from one thinking mode to another. Some of the problems highlighted by researchers, in particular, Piaget and Inhelder (1956), Martin (1976a, b), and Geeslin and Shar (1979) demonstrated the difficulties of basing a curriculum on the hierarchical structure of geometries. It is the opinion of this investigator that attempts to categorise inevitably break down and more importantly fail to deal adequately with children's whole assemblages of geometrical experiences. In addition the focus should be away from geometrical structures and towards a development of children's cognitive and conceptual frameworks. If productive learning is to be achieved in the classroom a geometrical curriculum should take into consideration the different ways that individual children perceive the tasks. Such perception is influenced by factors like ability, previous learnings, and stages of development. Different children will take different routes in organising their geometrical concepts.

One generalisation is made. From the research already carried out it appears that children are more comfortable with projective rather than euclidean ideas. Certainly they prefer projective transformations to enlargements, reflections, rotations, and piecewise congruent transformations. As evidence of this, an account of such preferences is contained in chapter three. In the light of these and other findings about children's interpretations and uses of line drawings, certain modes of children's geometrical thinking are presumed.

Originally it is suggested that, in order to develop geometrical concepts, certain specific modes of geometrical thinking are required. Children need to be familiar with the following types of activities:

(a) topological perception of the environment;

(b) straightness and polygons;

(c) projective viewing;

(d) affine representation;

(e) visual perception (visual illusions) and

(f) co-ordination of views.

As the research progressed this analysis appeared more and more inappropriate. The inadequacy of categorisation and the difficulties of encapsulating themes under brief heading became obvious. The framework began to collapse. The formative nature of much of the evaluation led to a reappraisal of the list and it was revised and restructured as follows:-

(a) perception of the environment;

(b) viewing and visualisation;

(c) affine representation; and

(d) projective co-ordination.

Later still these were subsumed under the unifying theme of a general projective perception.

Perception of the Environment. Under the heading "topological perception of the environment" several activities were produced which involved children in attempts to depict their thinking about views of their immediate environment. Their responses were not always purely topological or even partly so. It was decided to omit the word "topological". This had its difficulties, because it could be argued that the whole of geometry could come under environmental perception and an alternative title might be "How children perceive their environment" but after due consideration the title above was retained as being more appropriate.

It is a point of interest that the topological diagram of the London underground (often called amusingly "the underground map") uses a transformation of curved lines to straight. So it appears for simplicity that a projective invariant is used in an essentially topological exercise! Some children employ a similar strategy when they represent a meandering path through a wood by a pair of straight parallel lines. Analysing perceptions of the environment is not easy!

Viewing and visualising. This is the title given to the combination of "projective viewing" and "visual perception". "Projective viewing" emphasised the mapping of objects onto a plane, the intention was to focus children's attention to the mechanics of such mappings. "Visual perception" was the title given to the visual illusions and related themes. These two kinds of activity were found to be closer together than would have been expected. Under the revised title of "viewing and visualising", children's activities and responses are combined. They involve seeing scenes and objects from a particular viewpoint. This mode of thought links with affine representation and with projective co-ordination.

Affine representation. Mathematically this representation is the one obtained by viewing from an infinite distance (with the object still visible!) Parallels in the original figure or sketch remain parallels in the transformed representation. Shadows cast by the sun are good examples. Plans, elevations, and other conventionalised representations such as isometric drawings depend on the use of an affine (rather than a more general) projective transformation. As such they are not what is seen by viewing.

Therefore this requires a different approach. It entails an artificial and abstract mode of thought which is not necessarily needed for viewing or visualising.

It may not be necessary to point out to children that plans and maps are not views from above. Such an action was discounted in this research and the description used for plans was that of "views from above". This approach caused children few problems. Nevertheless, for a true plan an affine representation is essential. It has been pointed out in chapter two that perspective and projection depend on two very different premises. They have distinctively different underlying structures. A map is not a view from above though it may be a good approximation to it. Visual illusions often depend for their effectiveness on the use of an affine representation. This has been analysed in the fourth chapter. .However, the differences between an actual view and an affine representation remain. They require different ways of thinking. Both modes "viewing and visualisation" and "affine representations" are forerunners to another mode, one in which a change of viewpoint is required.

Projective co-ordination. The final set of activities are based on co-ordination of views. These can, and do, vary from the very easy to the exceptionally difficult. Putting oneself in one's imagination into a different place is difficult. Visualising the view from there presents major difficulties. These are not unconnected with plans, maps, and visual illusions. Again this modes of thought merges into others. It does, however, require a different kind of thinking activity and hence is kept as a separate classification.

This discussion suggests the possibility of an underlying anarchy. The interdependence of various modes of geometrical thinking appears to be more nearly the real state of affairs than unique categories.

Such an anarchy is difficult for most teachers. Some organisation is essential however simplistic it might be. This study proposes an organisation based on a projective approach and appears to be the best way forward.

Straightness and polygons. This section provides a projective approach to straightness. It is an attempt to introduce the concepts of planes, straightness, and straight line figures, as projective invariants. The exercises hint at the projective equivalence of all triangles. Triangles are seen as bounded by three line segments rather than the joins of three points. Flatness and straightness do not constitute a mode of thought but have obvious connections with other geometric thinking modes. They form a basis which underlies the usual euclidean geometry.

A general projective perception. This study advocates a projective approach to all aspects of geometrical learning. It suggests that a general projective perception is needed. This would emphasise to the primary teacher that:-

(a) there is a need to be continually concerned about children's perceptions of spatial ideas;

(b) an awareness of the value of considering projective approaches to geometry is necessary;

(c) it is important to be aware of geometrical modes of thinking and how these modes are inter-dependent.

The Significance for teachers and learners

The modes of geometrical learning, detailed above and subsumed under a general projective perception, have great significance in the classroom. They emphasise the needs of children and they provide foci for action.

Teachers are encouraged to have an awareness of children's geometrical perceptions. Questions may be asked such as: how are children perceiving the environment?; what do children view and how are they visualising representations of objects and scenes?; can they cope adequately with, and make sense of, certain affine likenesses?; what are their thinking processes when presented with a variety of viewpoints?

The thinking categories give teachers a perceptual and conceptual organisation upon which to base their teaching strategies. Thus exposition, questioning, discussion, display, and dialogues with individuals may be enriched. The projective nature of the modes also enables teachers better to determine aims and objectives. Their goals can have a projective flavour which makes geometrical learning more relevant.

By drawing attention to the nature of "seeing", (how, what, and from where) geometry is made more relevant and less isolated from other parts of the curriculum. The modes of geometrical thought cross subject boundaries and are intended to provide aspects of an integrated programme. Geometry may then become, in a more relevant way, a distinctive and more vital part of children's conceptual development. They should guide teachers into appreciating some of the complexities in the problems involved in guiding children's geometrical and spatial problems. Hence children's spatial-perceptual abilities should improve.

As an example, "viewing and visualisation" may be taken as a mode of thought, as an active focus. A curriculum package of activities is provided under this heading. Such activities enable the primary teacher to select from a variety of approaches. Exercises, puzzles, and questions all aim at motivating children into thinking about what "seeing" involves. The other modes are similarly organised.

This approach, based on developing projective modes of thought, enables teachers to direct and children to acquire learning strategies. These strategies are far less likely to occur in a more traditional approach.

Possible Avenues for Further Research

There is a sense in which all formative evaluations are necessarily incomplete, that is, there always appears room for improvement. A common task for all teachers is to organise learning tasks which are judged to be appropriate to individual children in a group, class, school, or district. All curricula, no matter how well devised, have to be interpreted. The curriculum developer has to recognise this fact and therefore, at some point in time, the evaluation, restructuring, and modification of a piece of work has to cease. A final version has to be prepared if the work is ever to be used. The present study is, of course, no exception. It would be nonsense to suggest otherwise. Indeed it is the investigator's intention that improvements should be made by teachers to suit their particular circumstances. They should be encouraged to modify the exercises in ways which seems to them to be appropriate. In particular it is hoped that teachers would substitute their own specific and more immediately relevant exercises for the more general ones provided in this study. Photographs and sketches of children's own homes and local area should be used whenever possible. For example, a visit to a place of interest may be arranged in connection with an unmathematical curriculum aspect. Plans and photographs of the place are likely to be available. Aerial views may also be obtained. With this information and from the children's experiences of visiting the site a two or preferably three dimensional model may possibly be constructed.

The style of the any curriculum package is always open to improvement.

In this case further attention could be given to the layout, the way various aspects are organised - for example, exercises, highlights, recapitulation, - and generally to the task of keeping children interested in the exercises. The language and vocabulary could be reconsidered, particular attention being paid to ensuring that children can read and follow the given instructions.

It may be that children could profit from different styles of instruction. To research this, a few typical activities could be provided in alternative forms. One form might consist of a minimum use of instruction, another might use a more discursive style with both redundant and recapitulative material (rather like that used by a teacher talking to a class or group of children). An analysis could then be made of children's responses, though this would need careful thought as children may be unfamiliar with one or other style of written mathematical English.

The use of colour in the text might be examined - for example, different colours could be used for different types of writing, exposition, instruction, example, highlights, definitions, and so on. Different types of printing could similarly be investigated. Colour could also be considered for use in the line drawings and photographs perhaps providing extra insight or motivation; and the motivational aspects of using photographs rather than, or as well as, line drawings could be made - again, possibly, by using the same exercise with alternative pictures. Care would be needed here to take into account such extra information such as depth cues which might perhaps be provided by a photograph rather than a line drawing.

In a sense these considerations are merely adjustments to the scheme, perhaps cosmetic or matters for further research. A more pressing consideration is the problem of incorporating such a curriculum scheme into what many consider to be an already crowded curriculum. This projective curriculum will only be used if it is seen to be more relevant and more useful than that which exists. Part of its usefulness is the comparatively little extra time that the use of this package entails. Most of the activities fall within areas which are not usually considered mathematical. Timing is important when presenting learning experiences. This package could be spread over four or five years of schooling and require not more than five to ten hours in any one year, where it might replace or enhance the normal curriculum.

The problems of innovation in schools are well known. The implementation of the material depends on its being seen as a valuable resource in the classroom and to some extent on the degree of teacher commitment. The problem for the developer is to make the package different enough and attractive enough to be acceptable. At the same time it needs to be "teacher proof", to the extent that a teacher who is not Erlangen orientated may use it effectively.

One large research activity, with which I hope to be involved in the future, is to evaluate the use of such projective activities against a projective free curriculum. Such a concern has been part of this study but has not been developed further, it is a second stage study. What needs to be considered now is the proper place and context for introducing angle and length thereby incorporating the projective approach into the geometry of shape and size. Or, as I prefer to consider it, incorporating this geometry into a wider, more satisfying conceptual projective framework.

Further consideration should be given to the place of geometry, and projective geometry in particular, within the whole curriculum. This would involve a conbined research activity involving other disciplines and interests.

Very exciting possibilities exist now that microcomputers are generally available. A great deal of research is needed to consider valuable ways of improving children's learning using microcomputers as additional resources. In the context of a projective curriculum the advent of graphics has considerable potential.

A simple example might be giving an affine and a projective view of a cuboid. By pressing a few keys the two views may be exhibited alternatively or simultaneously.

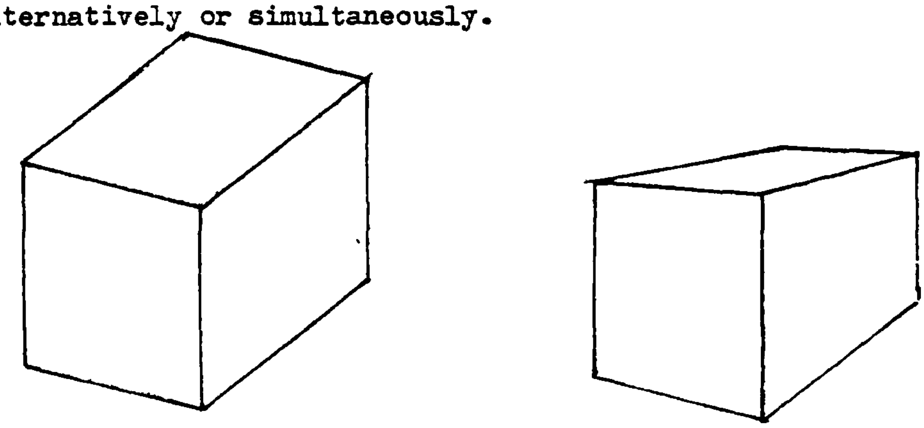


Figure 26 An affine and a projective view of a cuboid

Children could be asked to draw or copy these. Comments and discussions could be initiated, focusing on comparing and contrasting the line drawings. This is however a simple approach and might well be achieved better with line drawings on paper. Microcomputers may be used more purposefully.

Graphics already are in existence for showing continuous rotations of a cube in affine representation. It should not be difficult to extend the range of figures. It should be possible to set up a program so that a given affine representation can be transferred from paper to the microscreen and then modified by a program to simulate rotations in three dimensions. Such rotations could be under the control of the operator.

A further development would be possible if child operators were able to transfer their own diagrams onto the screen and then use the computer to obtain other views. This should lead to a better appreciation of co-ordination of views.

Exciting as these possibilities are, they hardly impinge on a perspective approach to geometrical experiences. The sort of program which I would wish to develop is that of showing continuous changes from one projective view to another. Ideally children should be able to transfer their own or other line drawings or photographs to the microscreen. They should then be able to control their viewpoint by simulating a rotation about some axis. A print out could then be obtained for different views of an object or scene. Such views may be projective, affine or euclidean. The effect on such exercises as Which is the tallest building? (C10) should be dramatic. The projective view may be turned into an affine view, the view point be rearranged so as to be in front (if necessary) and the tallest building would be immediately obvious. A projective line drawing of a scene (a village, a local view, a building, or similar) would be transferred to the screen, the viewpoint changed and a print out obtained of the scene from another vantage point. Shadow experiments and visual illusions may similarly be organised dynamically. The interconnections between a view and a plan, elevation, or map could be enhanced by taking the viewpoint further away or nearer. To sum up, the impact of a projective approach to geometrical experiences might be radically and dramatically reorganised by the use of suitable computer graphics. With or without such graphics an emphasis on projective ideas is needed.

The investigator believes that primary schools have for too long neglected suitable projective experiences. These can be of great value to teachers and to children in their care.

A projective element in the curriculum would:-

(a) help link different subject areas;

(b) enable geometrical investigations and problem solving to proceed more efficiently;

(c) broaden and enrich geometrical perceptions;

(d) give a conceptual clarity to spatial relationships;

and above all

(e) bring a new vigour and dynamism to the learning of geometry.

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APPENDIX ONE

A Projective Geometry Curriculum for Children in Primary School

Perception of the Environment

Number	Title
A 1	My way to school
A 2	My home
A 3	Where are things?
A 4	What can you see?

Straightness and Polygons

Number	Title
B 1	Which path?
B 2	Straight
B 3	Which parts are straight?
B 4	Flat
B 5	Joining points
B 6	Plane surfaces
B 7	Joining more points
B 8	Regions
B 9	Triangles
B 10	Triangles and quadrilaterals
B 11	More triangles, quadrilaterals
B 12	More regions
B 13	Yet more regions
B 14	Constructing quadrilaterals
B 15	Drawing straight lines
B 16	New lines from old
B 17	More new lines from old
B 18	Special noughts and crosses
B 19	Special line noughts and crosses

APPENDIX ONE

Projective Viewing

Number	Title
C 1	Which is the same?
C 2	Which are drawings of the same?
C 3	Odd one out
C 4	More odd ones out
C 5	Shadows from a lamp
C 6	More shadows from a lamp
C 7	Yet more shadows
C 8	Skeleton drawings
C 9	Drawing through a window
C 10	Which is the tallest building?
C 11	Which is the tallest
C 12	Viewing through a paper frame

Affine Representation

Number	Title
D 1	Which are views from above?
D 2	Which drawing looks most like?
D 3	Views from above (plans)
D 4	Drawing plans
D 5	Which is the plan?
D 6	Front views
D 7	Which is the front view?
D 8	Solids from drawings
D 9	Models from drawings and plans
D 10	Using more plans
D 11	Which drawing looks most real?

APPENDIX ONE

Visual Perception

Number	Title
E 1	Seeing things
E 2	Looking at things
E 3	Looking at more things
E 4	Lines converging at a point on the horizon
E 5	Inserting missing objects
E 6	Where is the middle?

Co-ordination of views

Number	Title
F 1	Where are they?
F 2	Making a drawing look right
F 3	Tiled floors and chessboards
F 4	More chessboards
F 5	More drawings and paintings
F 6	Where are you?
F 7	Which drawing is from which place?
F 8	Making more drawings look right
F 9	Drawings of large cubes
F 10	What will it look like from other places?

A1

MY WAY TO SCHOOL

Unstructured drawing - to be used diagnostically.
(Perception of the environment.)

AIM.	To discover how children perceive their immediate environment.
PROCESS	Drawing or Painting.
MATERIALS	Pencil, crayon, brush and paint, charcoal.
WHAT TO DO	Draw a picture of "My way to school".
MOTIVATION	Children should be given brief instructions of what they are expected to do. Perhaps they could close their eyes and imagine themselves on the way to school. They may be asked about the process and materials they wish to use. Their journey should fill the sheet of paper.
LEARNING	To realise that their journey is a continuous one - mathematically speaking - with no "breaks" in the path. To consider relationships of one part of the journey to another. To take account of changes of direction, corners, curved parts of the journey, particular points of interest on the way.
DISPLAY	Of some of the drawings.
DISCUSSION	Ask children to look at a particular journey. What do they notice? What special points does this picture show? Is the material good at showing this? Why is this shown like this? Do not be very critical, the object is to find out how they see things.
VOCABULARY	Positional and relational words e.g. connected, path, over the road, round the corner etc.
ENRICHMENT	Some one wants to get from school to your home. Say how they would do so. Use your drawing. Write down how you would tell the person to get there. Remember they have to go straight from school.

A1 ctd

MINIMUM EXPECTED OF CHILDREN	A reasonable attempt within their abilities to show the journey.
EVALUATION	Use the exercise as a means of obtaining feedback about their views. Is their journey continuous? Are they connecting the roads in the correct order? Are they taking account of corners, curved parts of the journey and so on?

A2

MY HOME

Structured drawing - to be used diagnostically.
(Perception of the Environment)

AIM To attempt to obtain drawings which "look right".

PROCESS Drawing, Painting.

MATERIALS Pencil, crayon, brush, paint.

WHAT TO DO Draw a picture of your home.

MOTIVATION Give brief instructions, say that you want a view of their home but not the usual one. Imagine they are outside their home on a fireman's ladder and they climb up and up. Try to imagine what it looks like from there.

LEARNING To realise the view will change as they get higher. To interrelate the various parts of the building. What will the roof look like? Perhaps they can see part of the side wall also.

DISPLAY Of some of the drawings.

DISCUSSION Ask the children to look at a particular drawing. Does it "look right" - say how difficult it is - how we are trying to learn what is right about a drawing. Do not be very critical, generate reasonable discussion.

VOCABULARY More positional and relational words e.g. joined to, on top, upright etc.

ENRICHMENT Ask some children to draw a picture of their home from in front, from the side, from above. Do the same for another object - say a table, matchbox, rowing boat etc.

MINIMUM EXPECTED OF CHILDREN A reasonable attempt to visualise their home from an unusual vantage point. An attempt to connect the various parts of the home in a correct manner.

EVALUATION Use the exercise for feedback on their thinking about the way their home is constructed. An attempt to make a wall look vertical for example.

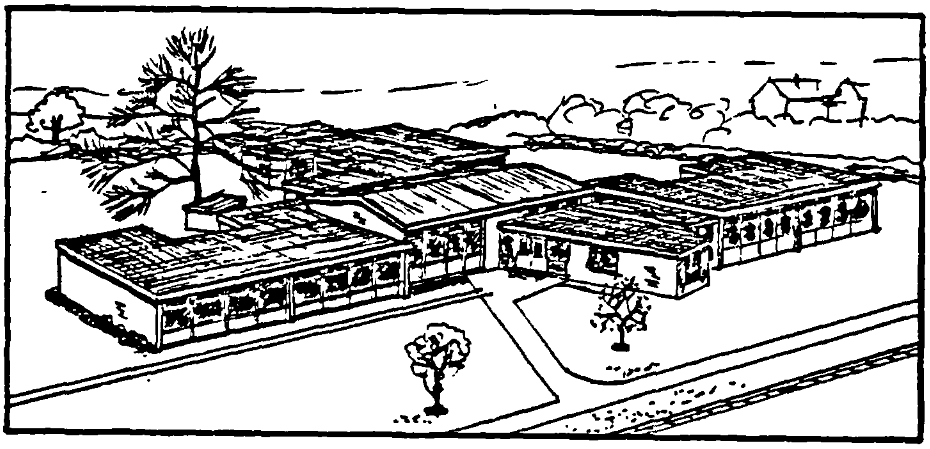
A3

WHERE ARE THINGS?

Discussion, use of language, spatial vocabulary.
(Perception of the Environment)

AIM	To provide children with opportunities to interpret projective views; to trace out paths and routes; to give instructions for going from one place to another. (As the picture distorts distances and angles the instructions are likely to be topological, that is of the sort "go down this path, turn left at the cross-roads and then go past the barn towards the ...". Distance and angle are unimportant in these interpretations.)
PROCESS	Tracing paths, giving verbal descriptions of routes, drawing of picture.
MATERIALS	Pencil, felt tip, paint, brush.
WHAT TO DO	Here is a school. A path leads to the entrance, from the hall you can look out on this path. Where do you think the head's room is? Why? Where is the hall?
DISCUSSION	One child comes out and describes the school. Where are the classrooms? The head's office? Where would the playing field be? What would help to make it clearer? A plan? What would a plan look like?
MINIMUM EXPECTED OF CHILDREN	An ability to make reasonable suggestions about the position of various parts of the school.

A3 ctd



A4

WHAT CAN YOU SEE?

Question and answer, leading to discussion and other activities as they seem appropriate.
(Perception of the Environment.)

What can you see in the drawing of (a) a farm? (b) a village or hamlet? (c) a factory? (d) a city centre? (e) a small estate? (f) a bungalow with the roof missing? (g) a derelict church?

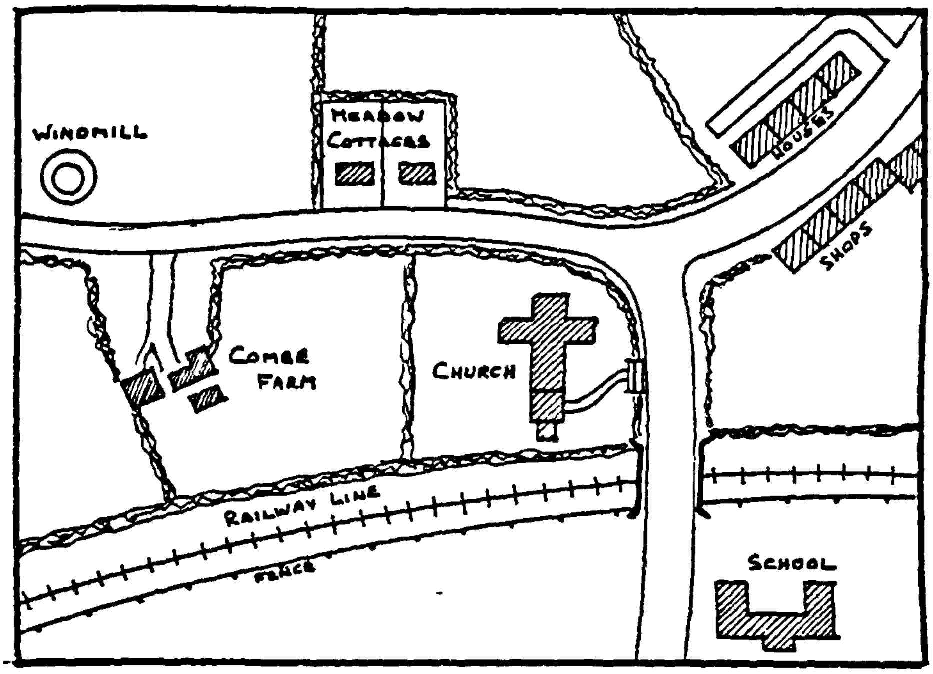
AIM — To promote question and answer to see how children view such perspective views.

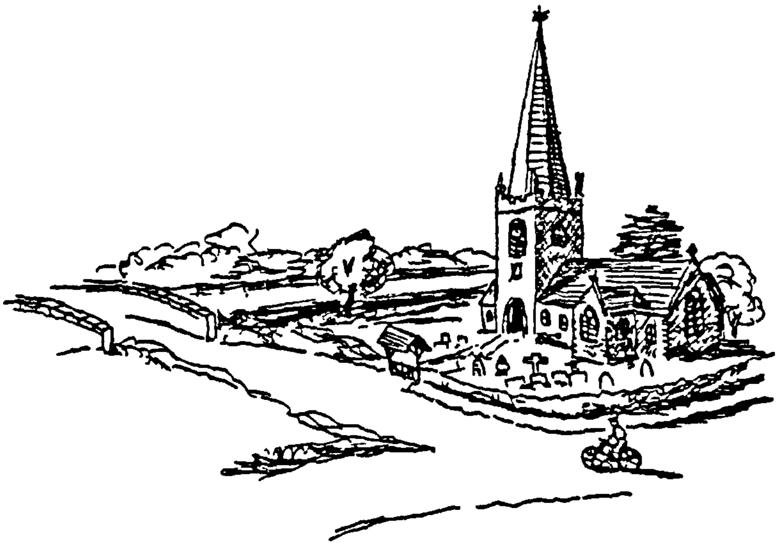
PROCESS — Teacher led discussion. Possibly drawing.

WHAT TO DO — (For example) Here is a picture of a small village. Point to the church, the school etc. Tell me how you would get from the windmill to the school etc. Where is the lych-gate? Point to the lych-gate on the drawing, to the bridge over the railway. Where is the school on the picture? Why? Put a cross on the map where you think the drawing was made.

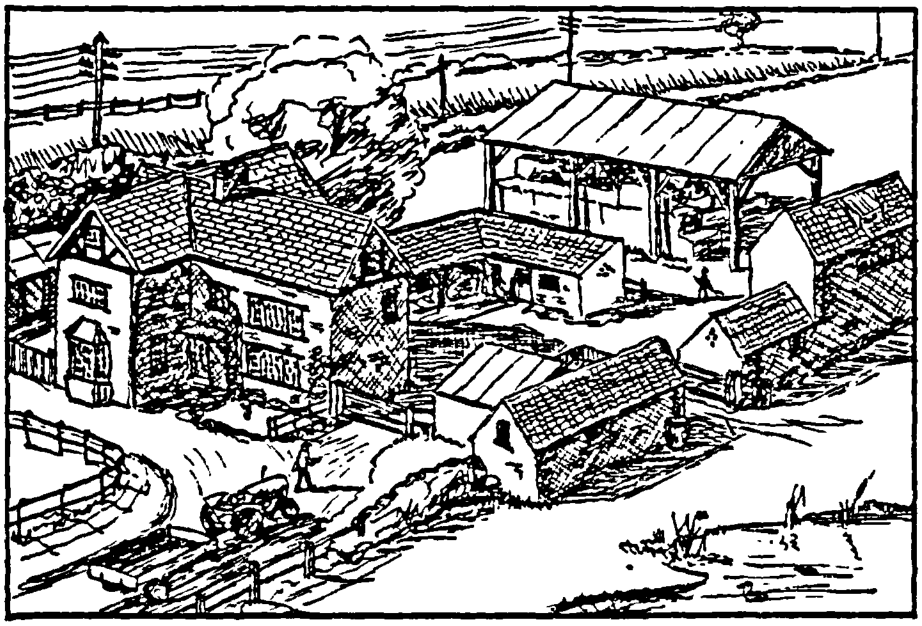
ENRICHMENT — (For example) A man is looking out of the church tower towards the windmill. Draw a picture of what he sees.

A4 ctd

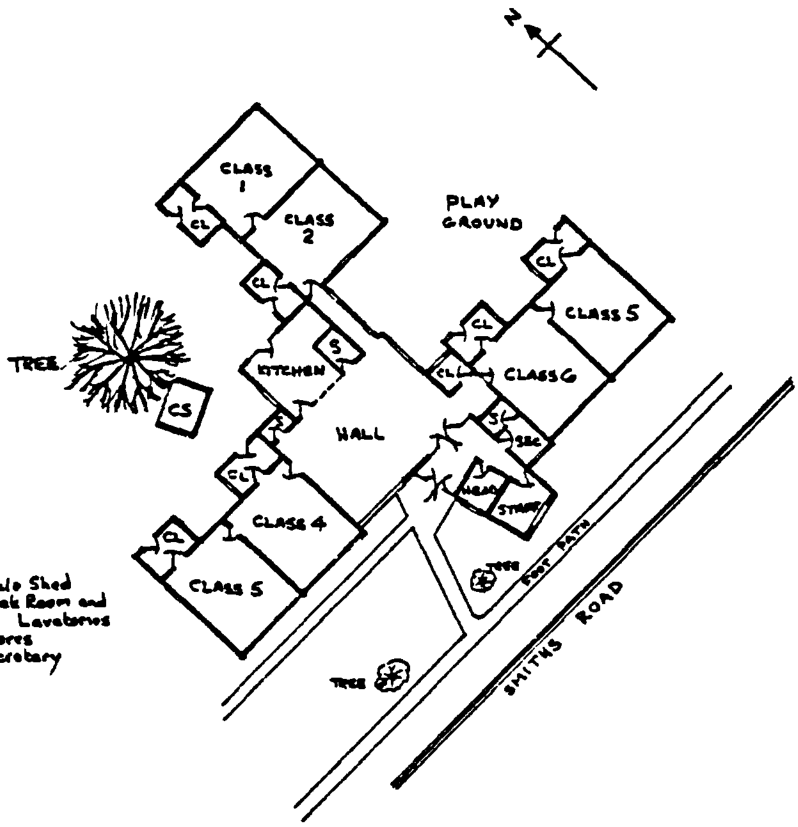

WINDMILL
MEADOW COTTAGES
HOUSES
SHOPS
COMBE FARM
CHURCH
RAILWAY LINE
FENCE
SCHOOL



A4 ctd

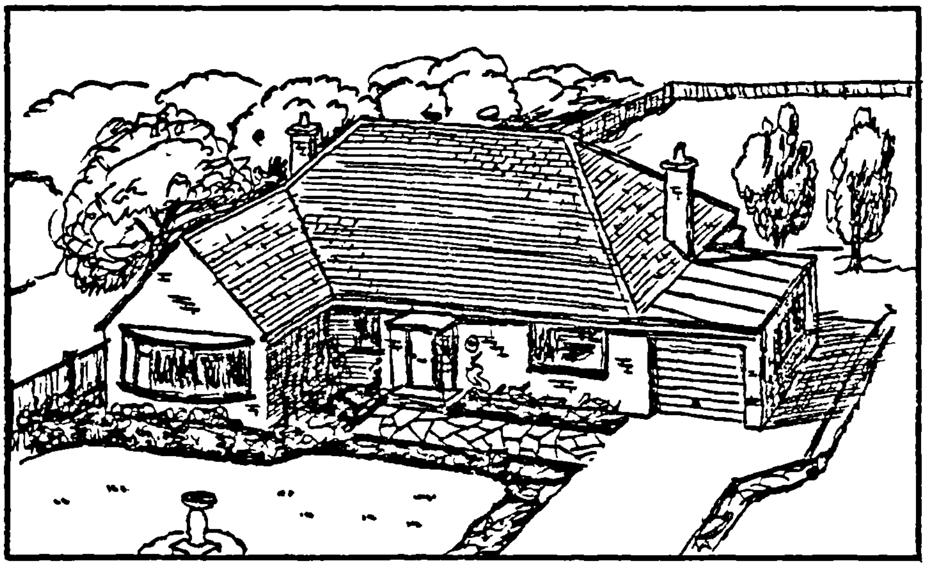


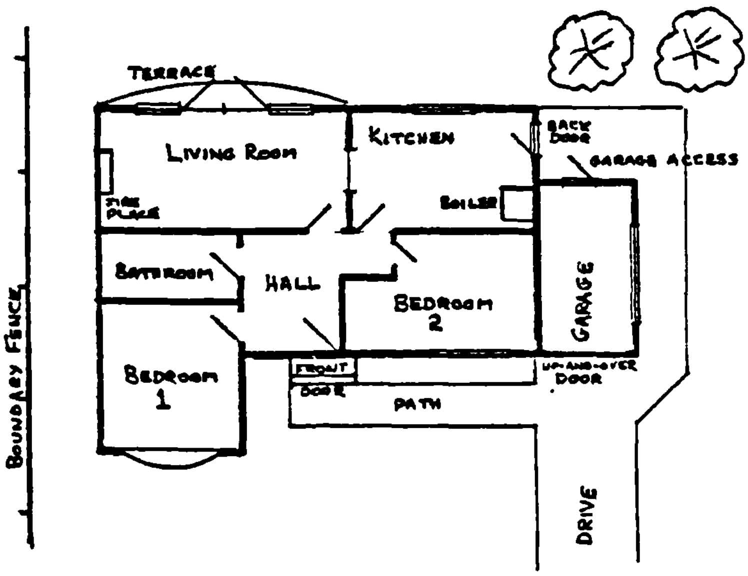
A4 ctd



KEY
CS = Cycle Shed
CL = Cloak Room and Lavatories
S = Stores
SEC = Secretary

A4 ctd




TERRACE
LIVING ROOM
KITCHEN
BACK DOOR
GARAGE ACCESS
FIRE PLACE
BOILER
BATHROOM
HALL
BEDROOM 2
GARAGE
BEDROOM 1
FRONT DOOR
UP-AND-OVER DOOR
PATH
BOUNDARY FENCE
DRIVE

B1

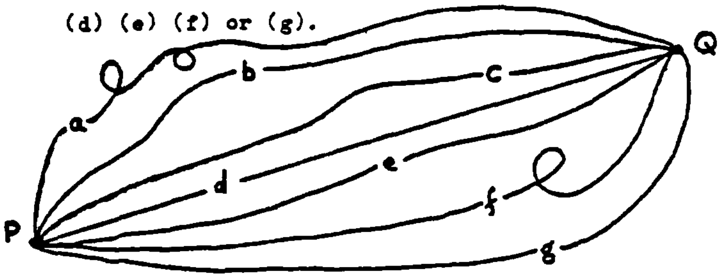
<u>WHICH PATH</u> ?

Multichoice questions, copying and completing. (Straightness and polygons)

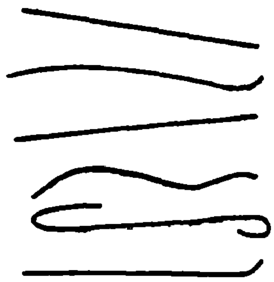
AIM Children will learn to distinguish straight lines from ones which are not straight, or not completely so.

PROCESS Individual work or use a class discussion with diagrams on the board.

WHAT TO DO A car can go from P to Q by way of the paths (a) (b) (c) (d) (e) (f) or (g).



1. Drive your car along each path.
2. Write down what is special about path (d).
3. Write down: Path (d) is called straight. The other paths are not straight.
4. Now copy and complete:



is straight
is not straight
is.............
is.............
is.............
is.............

COMMENT No doubt most children will have met the idea of straightness before, but some misconceptions may be apparent.

STRAIGHT

Drawing straight lines, discussion.
(Straightness and Polygons)

AIM To find out how children interpret the word straight.

WHAT TO DO Draw one straight line in each of these five square boxes. One line in each box. Make all your straight lines different.

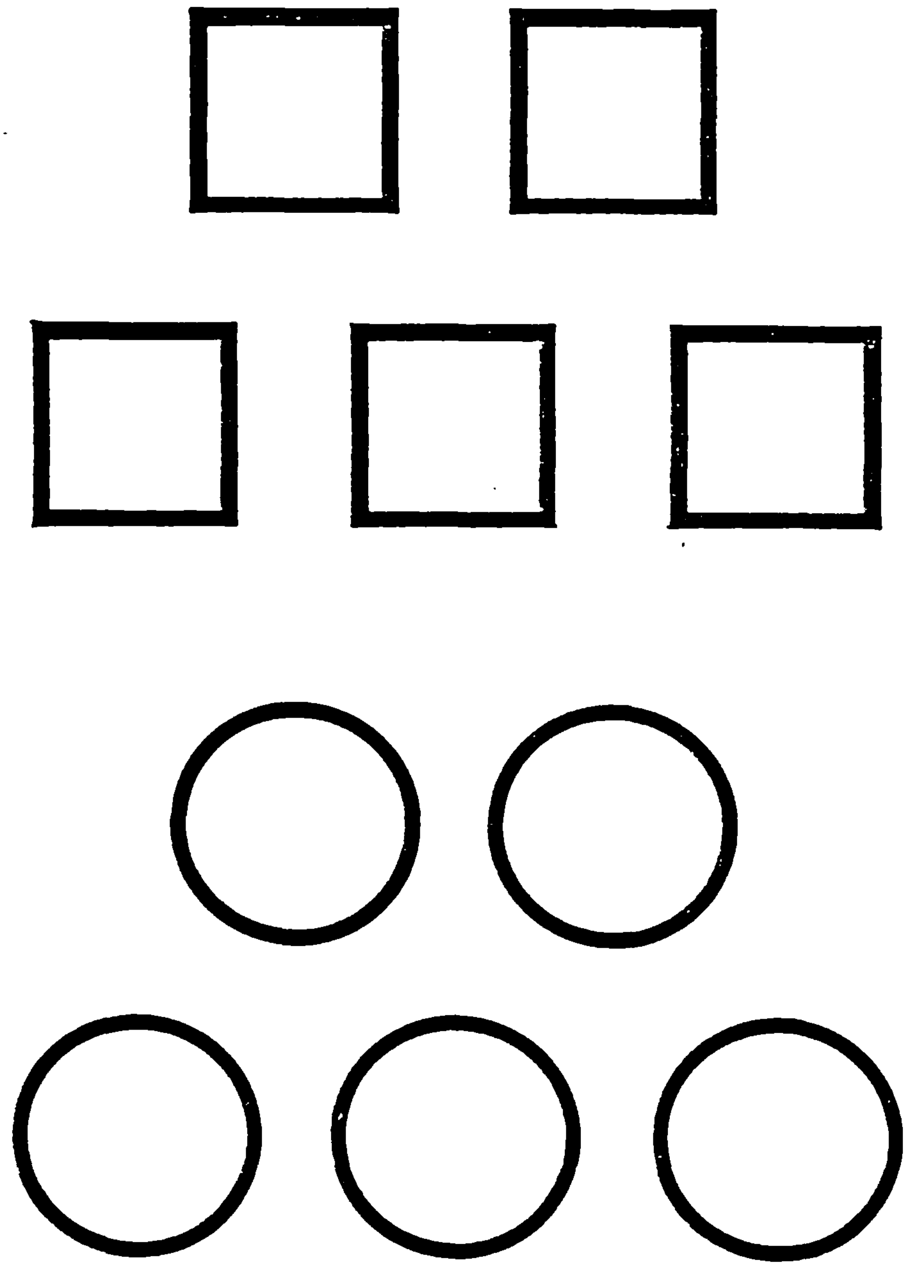
Now do the same with these circular boxes. Make all your straight lines different.

DISCUSSION Class discussion. Are all your straight lines really straight? Are they all — ?
Can I put in a line like this \ ? or this / ?
What is a straight line? How can we test if a line is straight or not?

VOCABULARY Straight, curved, line, wavy, direction, (and others)

EVALUATION See whether the surrounding shape, square or circle, influences their choice of straight line. See whether straight means horizontal or vertical but not sloping.

B2 ctd



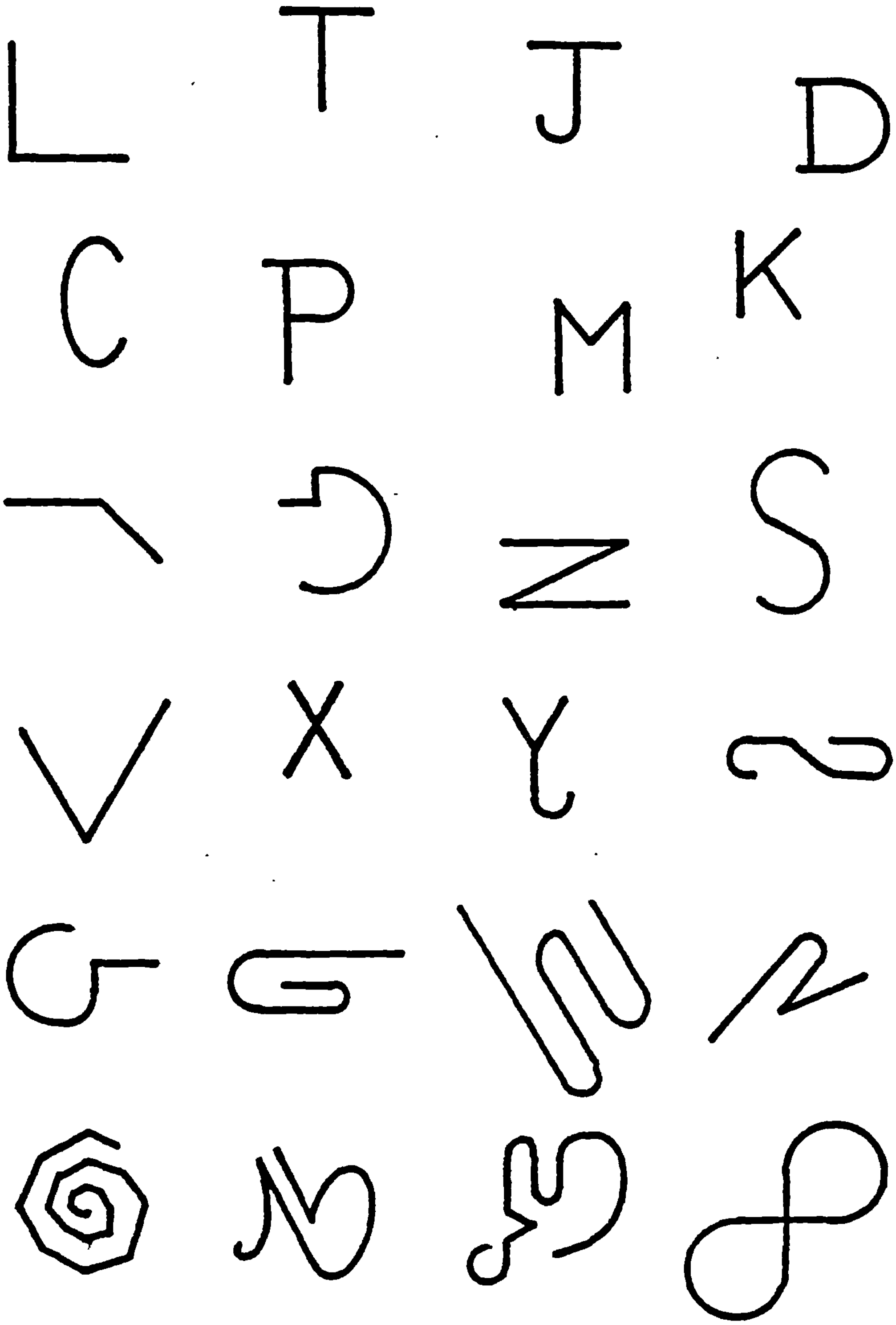
B3

WHICH PARTS ARE STRAIGHT?

Drawing straight lines, discussion, diagnostic.
(Straightness and polygons)

AIM	To find out which children consider sloping lines (i.e. not parallel to the edges of a sheet of paper) as straight.
MATERIALS	Crayon, felt tip pen.
WHAT TO DO	Here are some letters and shapes. Colour the parts of the letters which are straight. Do not colour the curved parts.
DISCUSSION	Why have some got different answers? Who is right? What is straight? Why are the later ones more difficult?
DISPLAY	If appropriate.
ENRICHMENT	Now make up some more and try them on your friends. Write about what you found out.
MINIMUM EXPECTED OF CHILDREN	Children coming round to the idea that oblique lines like and and are also straight.
VOCABULARY	Across, same direction, bending wavy and so on.
EVALUATION	See whether a child can distinguish between curved and straight lines independent of their orientation on the paper - this idea is met later.

B3 ctd



B4

FLAT

Question and answer, discussion. (Straightness and polygons)

AIM To provide children with opportunities to learn that the word 'flat' is used in a variety of ways, but that flatness in geometry is the absence of curvature on a surface.

PROCESS Discussion - the word flat is used in many ways, children may mention one or more of the following.

DISCUSSION When do we use the word 'flat'? Can you think of a time when you use the word? What is flat? What about the following? "Martin ran flat out" "Jane fell flat on her face" "I tell you flat you can't watch that" "After the visit to the zoo, Jim felt flat the next day" "Hold that tray flat, you'll spill it if it slopes" "The sea looks flat today" "That's a flat field" "Look at that block of flats" "I live in a flat" "Here are some pieces of wood - blocks, flats, longs and units."

ENRICHMENT Write some more sentences using the word flat.

DISCUSSION What about the following. "This floor is flat" "This wall is flat" Is a sloping roof flat or not? Why? Lead on to explain that drawing and painting and pictures usually are made on flat surfaces.

VOCABULARY Flat, even, smooth, straight, slope, sloping, wobbly.

EVALUATION The child should be able to sort out flat surfaces from curved surfaces. (whatever the orientation)

B5

JOINING POINTS

Activity, display. (Straightness and polygons)

AIM — Children will learn to fold a piece of paper to obtain a straight line.

PROCESS — Construction and testing.

MATERIALS — Pencil, crayon, sheets of paper or newspaper.

WHAT TO DO — Mark two points on a piece of paper like this  or like this  or like this 
Join up the points with a straight line using a pencil or a crayon. Try to make the line as straight as you can, but do not use a ruler. Now fold the paper so that the fold goes through both the points. Take a lot of trouble to get the fold in the right place. See how straight your line was.

DISCUSSION — Why is it difficult to draw a straight line? What is special about a straight line? Why is it difficult to fold it through two points? Can we practise this?

DISPLAY — Some of the attempts - with heading "Joining Points."

ENRICHMENT — Fold and flatten the sheet of paper to get lots of lines - how many can you get?

MINIMUM EXPECTED OF CHILDREN — The children can fold a piece of paper through two marked points.

VOCABULARY — fold, through.

EVALUATION — The child should consider that a fold gives a straight line independent of its position on the paper and that straightness is independent of edge of paper

B6

PLANE SURFACES

Activity - discussion. (Straightness and polygons)

AIM	Children will learn to call flat surfaces planes and to test for planes with one or both of the 'glasspaper' and 'ruler' tests.
PROCESS	Testing.
WHAT TO DO	Place two flat surfaces together - such as two hardback books - so that there are no gaps between. Move the surfaces over each other.
DISCUSSION	There should be no gaps showing. Find out what happens when one of the surfaces is not flat, not a plane. We can use this idea for making a piece of wood flat and smooth, using a piece of glass paper. Find out how to use a ruler to check for flatness. Why is it necessary to move the ruler around and up and down to test for flatness?
ENRICHMENT	Find some shapes which the ruler fits one way but not others (cylinder, cone) and ones which the ruler never fits (spheres, rugby ball shape and others)
VOCABULARY	Flat, plane, glass paper, even, smooth.
EVALUATION	The child should be able to distinguish between plane and curved surfaces.
COMMENT	Description of tests for flatness may be suitable for some children. For example: "There are two types of testing for flatness. We used a ruler to ...."

B7

JOINING MORE POINTS

Activity, testing for straightness. (Straightness and polygons)

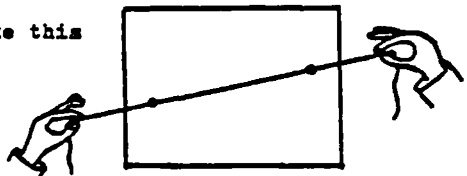
AIM Children should further develop the concept of straightness by testing with: (a) A piece of string, (b) Another piece of paper, (c) A ruler, and (d) sighting along the line by eye.

PROCESS Construction and testing.

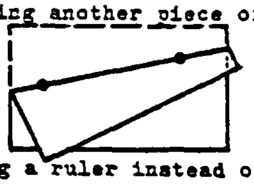
MATERIALS Pencil, crayon, paper, newspaper, string, (or cotton).

WHAT TO DO Mark two points on a sheet of (news)paper. Join up the points by a straight line or a line as straight as you can manage.

Test your line by holding a piece of string tight like this

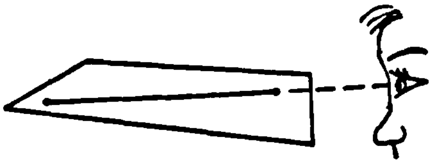


and by folding another piece of paper like this



and by using a ruler instead of the folded paper.

Try also looking along the line like this



MINIMUM EXPECTED OF CHILDREN The children can test the line using one or other of these methods.

EVALUATION The child should be able to test object in the room for straightness using string, folded paper, ruler or line of sight.

B8

REGIONS

Activity - possibly discussion and/or display.
(Straightness and polygons)

AIM Children learn to distinguish between various regions on a plane or curved surface.

PROCESS Construction of lines by folding.

MATERIALS Paint, crayons, pencil.

WHAT TO DO Fold a piece of (news)paper, then unfold and place flat. Crayon or pencil along the line of the crease. The fold partitions (separates, divides) the paper into two regions.

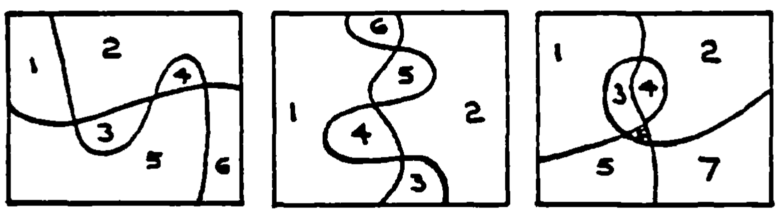
Now fold the paper again in a different direction and mark the crease as you did before. How many regions are there now? Colour or paint each one a different colour.

Take another piece of (news)paper. Mark two <u>curved</u> lines on it from one side to another. See how many regions you have now.

ENRICHMENT Do the same again, using curved lines, but make the number of regions greater - say six. (It can be done!)

VOCABULARY region, boundary, partition, separate, divide.

like this or this or this



B9

TRIANGLES

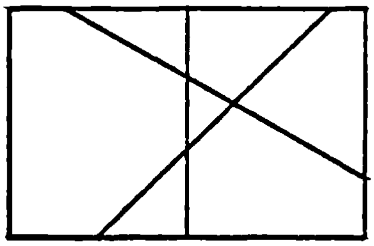
Activity, display. (Straightness and polygons)

AIM — Children learn to obtain triangles by three folds and that a variety of triangles may be constructed, base horizontal, base in other directions, different shapes - sides unequal - two sides equal, large angle, equilateral and so on.

PROCESS — Construction by folding.

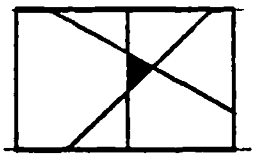
MATERIALS — (News)paper, crayon, felt tip, pencil.

WHAT TO DO — Fold a piece of (news)paper three times like this



On the paper write "Three folds gives $\boxed{}$ regions." (Fill in the box with the right number.)

Paint or colour in the middle region.



Take another piece of paper and tear off all the straight edges. Now fold three times again. How many regions now? Write the number on the paper. Paint or colour the middle region. This region has three straight edges. It is called a triangle.

ENRICHMENT — One fold gives two regions, two folds give four regions, three folds give seven regions, four folds, five folds?

MINIMUM EXPECTED OF CHILDREN — The child should be able to fold a variety of triangles.

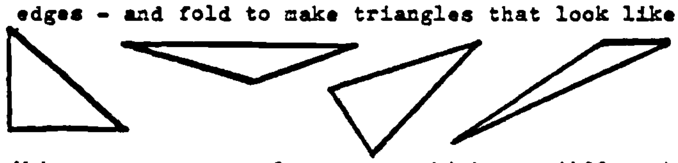
DISPLAY — An attractive display may be made of the painted or coloured regions.

TRIANGLES AND QUADRILATERALS

Activity, (Straightness and polygons)

AIM Children learn to obtain a variety of quadrilaterals by folding (and unfolding) a piece of paper four times.

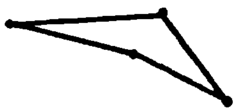
WHAT TO DO Use different sheets of (news)paper - torn into sheets about size A4 or a little larger, preferably with torn edges - and fold to make triangles that look like these:



ENRICHMENT Make up some more of your own which are different from these.

DISCUSSION Why are all these triangles? What is different about a triangle from other shapes? How is it they can be so different?

WHAT TO DO Use four folds and shade, colour or paint the middle shape you get now. See if you can make a square, a kite, a diamond, a rectangle, some other shapes.

Did you get  ?

LEAST EXPECTED Children can fold a variety of three and four-sided figures.

VOCABULARY Quadrilateral (or 4gon if quadrilateral is too difficult) (Polygons may be called; 4gon instead of quadrilateral, 5gon instead of pentagon, 6gon instead of hexagon etc. if so desired.)

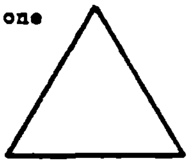
MORE TRIANGLES, QUADRILATERALS

Activity, (Straightness and polygons)

AIM To discover what the children have learnt about triangles, quadrilaterals and obtaining regions. This may be used as a test.

MATERIALS Pencil (and erasers!)

In the box draw a triangle different from this one



Draw a quadrilateral (shape with four sides or 4gon) which is unusual

Draw a quadrilateral (shape with four sides or 4gon) which is interesting to draw

Divide this box into seven regions using three straight lines

Divide this box into seven regions using two curved lines

B12

MORE REGIONS

Activity, (Straightness and polygons)

AIM	Children learn to construct triangles and quadrilaterals of various shapes with the use of a ruler.
PROCESS	Constructing straight lines using a ruler (straight edge)
MATERIALS	(News)paper, rulers, felt tips, pencil.
WHAT TO DO	Mark three points on a sheet of paper - make sure they are <u>not</u> in the same straight line. Use a ruler to draw a line through two of the points. Make sure you continue the line to both edges of the paper. Now join another two points (continuing the line to the edges of the paper). Now join the last two. State how many regions you have. Write "I have regions." Shade the middle region - on it write "The shaded figure is a triangle."
MINIMUM EXPECTED OF CHILDREN	An ability to draw a line through two points to a reasonable degree of accuracy.
VOCABULARY	Two lines meet in a vertex, each line is a boundary of the triangle. A triangle has three sides and has three vertices.

B13

YET MORE REGIONS

Activity, (Straightness and polygons)

AIM Children learn to construct more triangles and quadrilaterals.

MATERIALS Rulers, felt tips, pencil, (news)paper.

WHAT TO DO Mark 3 points to give an unusual or interesting shape.

Mark 4 points to give a four-sided figure - a 4gon or quadrilateral. Make some interesting or unusual 4gons - try making a square 
a kite  a diamond 
a 'caved in' 4gon  Remember they can be turned round like this  or this 

ENRICHMENT Make up some unusual shapes and give them special names. Can you make: an arrowhead, a shoe, a long thin shape, any unusual ones?

B14

CONSTRUCTING QUADRILATERALS

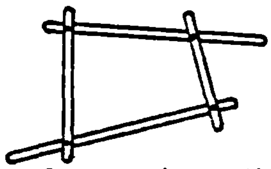
Construction (Straightness and polygons)

AIM Children learn to construct triangles and quadrilaterals or various shapes.

PROCESS Construction.

MATERIALS Thick card strips, instrument for making holes, paper clips, or Meccano and nuts and bolts, or straws and pipe cleaners, or wooden dowelling and elastic bands or orbit material.

WHAT TO DO Make a 4gon (quadrilateral) like this.



The hinges may be made with paper clips.

Can you change the shape without bending the strips and without breaking the figure?

Try this with a five-sided figure (5gon or pentagon) and with a triangle.

Which of these is rigid?

How would you make the others rigid?

VOCABULARY Rigid, 5gon or pentagon.

MINIMUM EXPECTED OF CHILDREN Children can make a quadrilateral and change its shape without breaking it. Children should realise that all triangles are rigid.

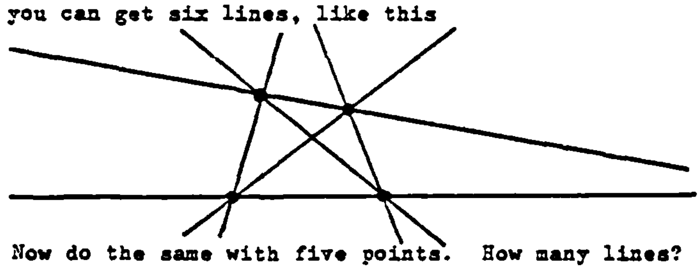
DRAWING STRAIGHT LINES

Ruler skills (Straightness and polygons)

AIM Children are encouraged to develop skills of drawing straight lines with rulers. They need to be told not to get their fingers in the way. To make sure that their lines go through the end points properly and so on.

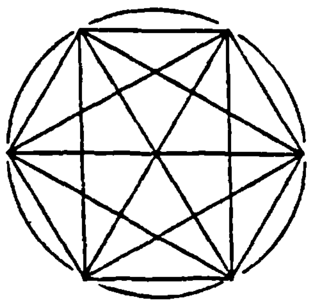
WHAT TO DO Put 4 points on a piece of paper like this

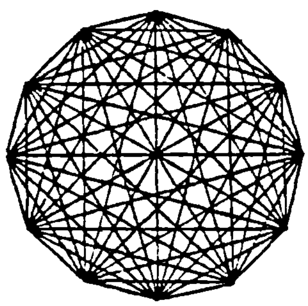
Join up two points - make the line go to the edge of the paper both ways. Do the same with another two points. If you use the points more than once you can get six lines, like this



Now do the same with five points. How many lines? Make sure your lines go properly through the points.

ENRICHMENT Draw a mystic rose - 4, 6, 12, or 24 points for example equally spaced round a circle - but do not extend the lines beyond the circle.





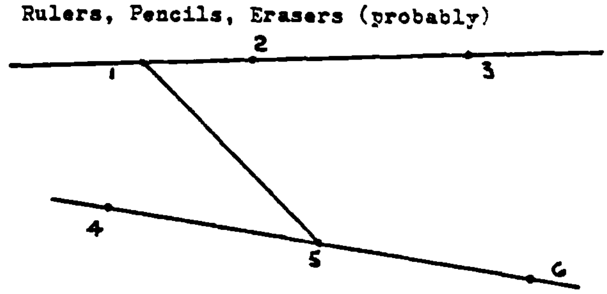
B16

NEW LINES FROM OLD

Ruler skills (Straightness and polygons)

AIM Children are further encouraged to develop ruler skills.

MATERIALS Rulers, Pencils, Erasers (probably)



WHAT TO DO Here are six dots, three one one line, three on another. Use your ruler to join 1 to 5 - this has been done for you already - and then to join 2 to 4. Call the point you get 7.
Use your ruler to join 1 to 6 and then to join 3 to 4, call the point where these meet 8. Be sure you have got the right point.
Use your ruler to join 2 to 6 and then 3 to 5. Call the point where these meet 9. Be careful again.
Write down what you notice about 7, 8 and 9. On your paper/in your book write "I notice that 7, 8 and 9......"
Try drawing two new lines and put three dots on these and see if the same result happens again.

MINIMUM EXPECTED OF CHILDREN Some children may have difficulty. They could try paper folding instead - but the ability to join two points by a straight line using a ruler is expected.

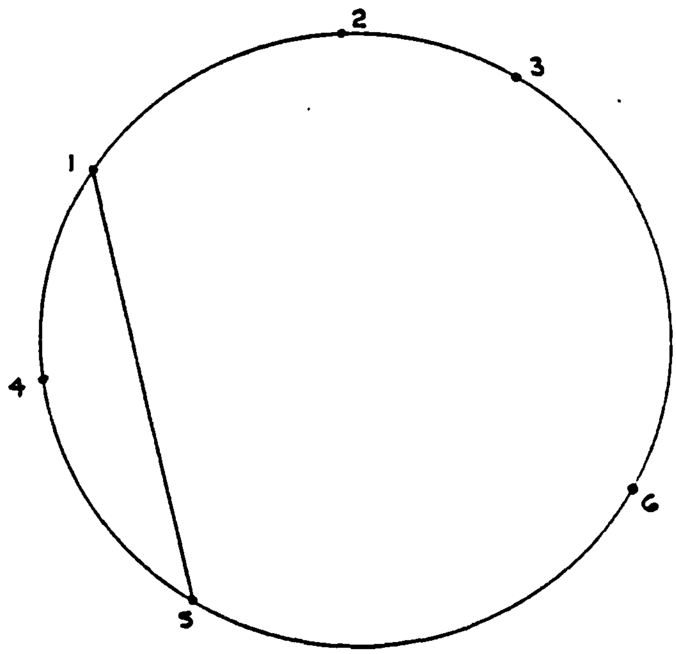
MORE NEW LINES FROM OLD

B17

Ruler skills (Straightness and polygons)

AIM Ruler skills.

WHAT TO DO As on NEW LINES FROM OLD



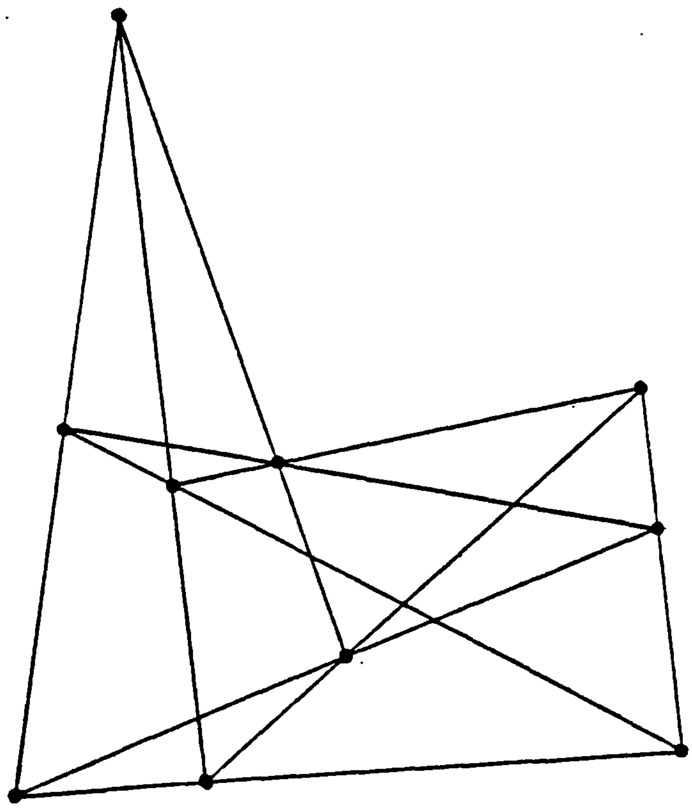
COMMENT The points labelled 7, 8 and 9 should still lie on a straight line.

B18

SPECIAL NOUGHTS AND CROSSES (TIC TAC TOE IN THE USA)

Two person game (Straightness and polygons)

AIM To provide children experiences in searching for three points in a straight line using a competitive game situation.



WHAT TO DO One player has small red discs - the other small blue discs. Take it in turns to put one of your discs on a blob. If you get three blobs in a row score 1 point. Carry on playing. If you do it again in the same game score another three points. Play a few games with a friend (or friends).

B19

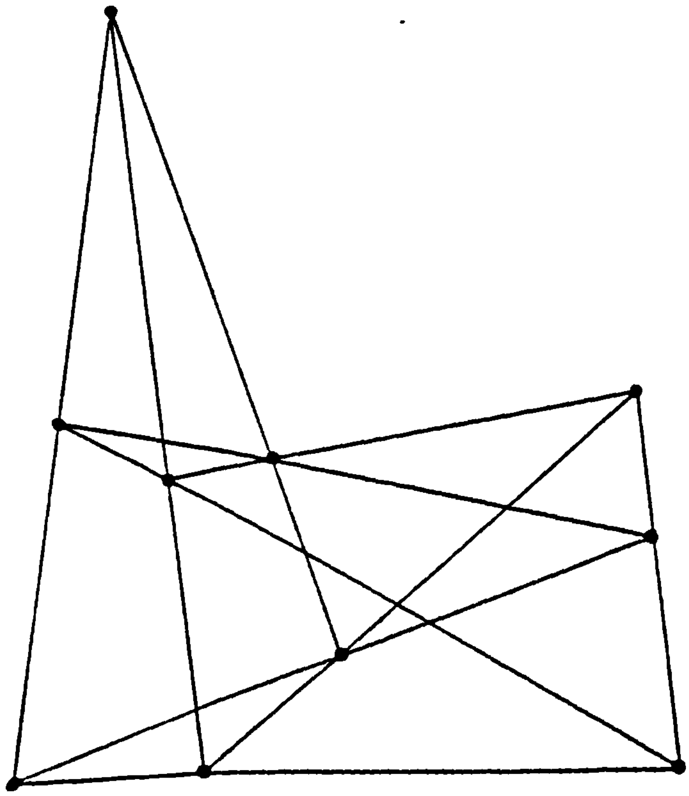
SPECIAL LINE NOUGHTS AND CROSSES

Two person game (Straightness and polygons)

AIM To provide children experiences in searching for three lines through a point using a competitive game situation.

MATERIALS Coloured pencils, felt tips (red and blue) Alternatively coloured transparent thin strips-may be constructed.

WHAT TO DO One player colours a line red, then the other player colours a line blue and take it in turns after this. Score a point if you have three red or three blue lines meeting at a blob. Carry on playing. Score a <u>further three points</u> if you do this <u>twice</u> in the same game. Play a few games with friends.



C1

WHICH IS THE SAME?

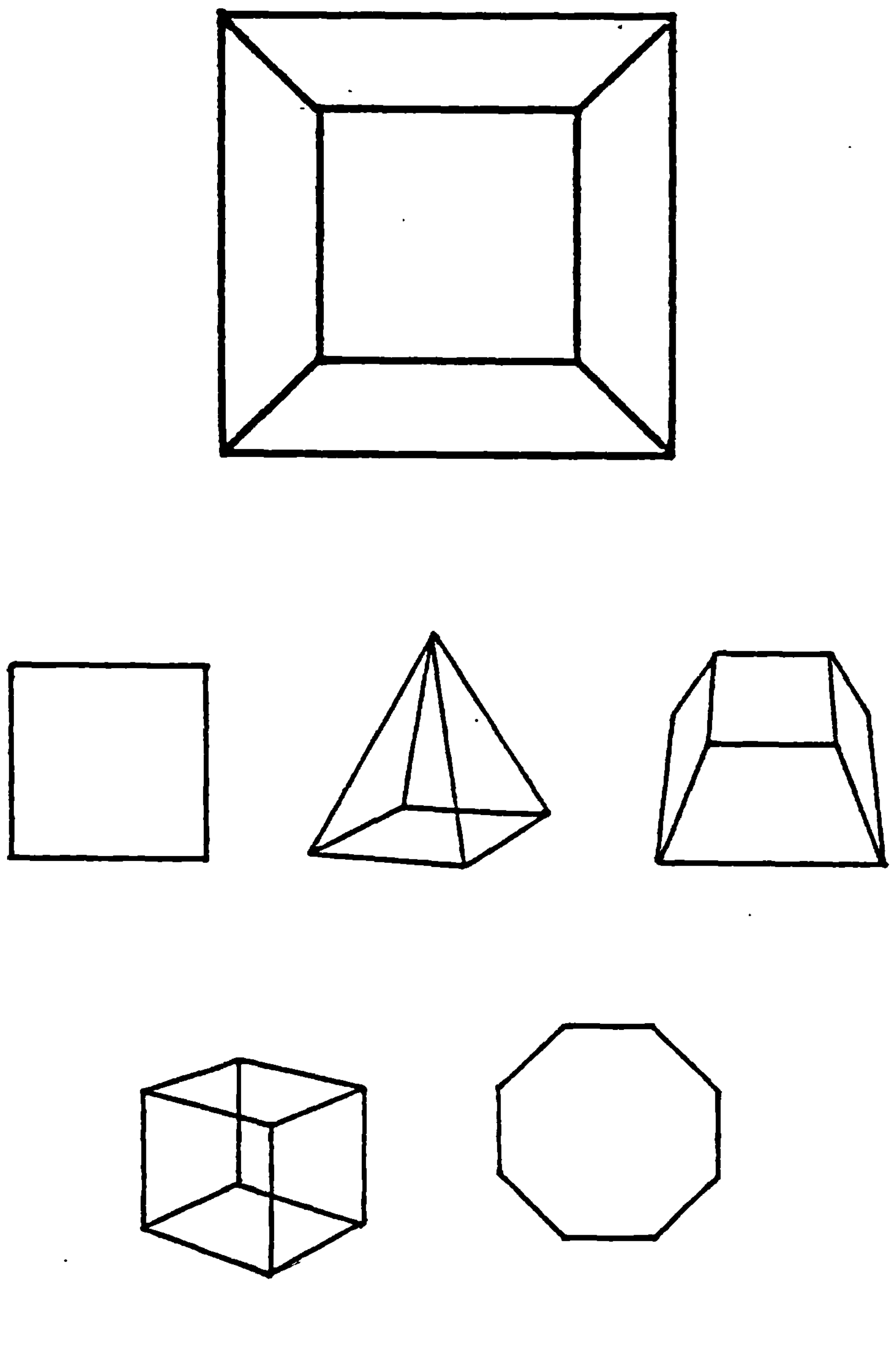
Multichoice diagnostic test leading to discussion.
(Projective Viewing)

AIM To provide children with experiences of drawings of three dimensional 'skeleton' figures.

WHAT TO DO Here is a model made from drinking straws. Below are five drawings but only one is a view of the model from a different place. Say which drawing it is.

COMMENT It may give useful information to ask children why they made their choice. Do they still think they are right? A useful discussion may develop from this.

C1 ctd



C2

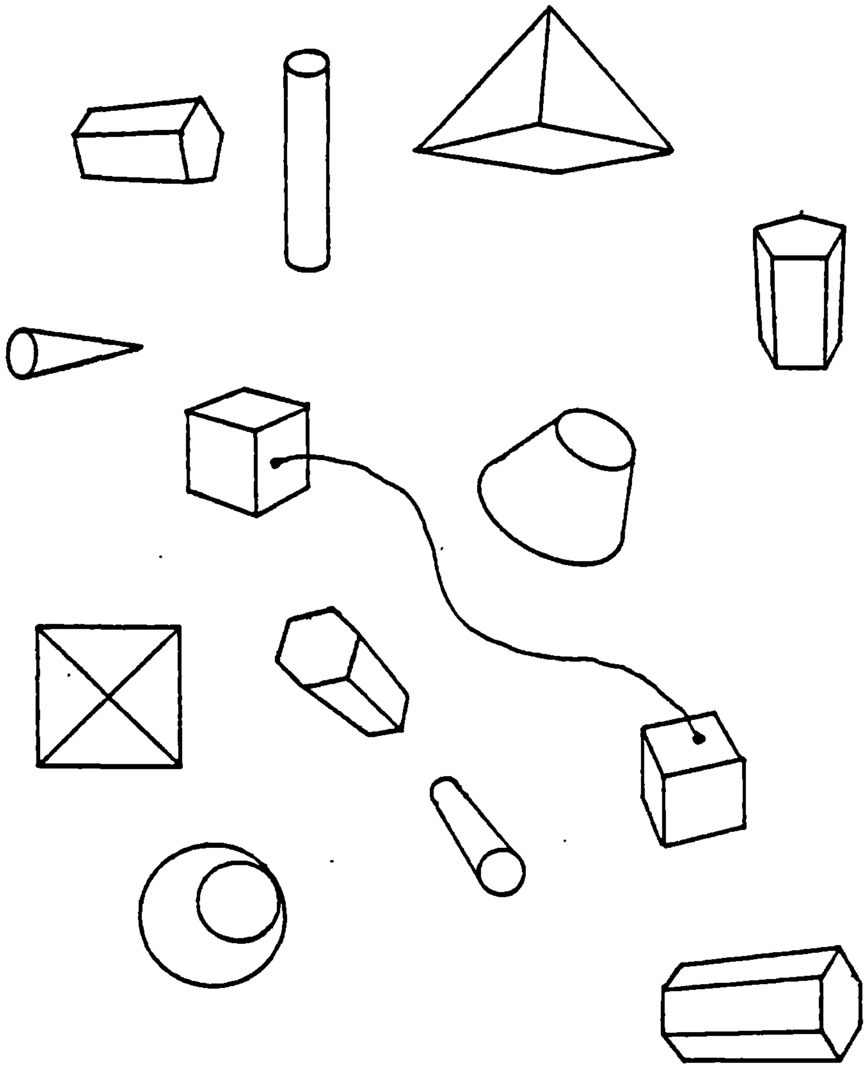
WHICH ARE DRAWINGS OF THE SAME?

Selection by matching. (Projective Viewing.)

AIM To find out which children are able to match different drawings of the same object.

WHAT TO DO Here are some drawings. Join the two drawings of the same object by a curved line. One has been done for you already. One drawing is an extra one and does not have a partner.

C2 ctd



C3

ODD ONE OUT

Multichoice questions. (Projective Viewing)

AIM To discover whether children can recognise different line drawings of the same object.

PROCESS Selection.

WHAT TO DO Read to the children:

"In the row labelled A there is an odd one out, one that is different from all the others, put a cross against this odd one out."

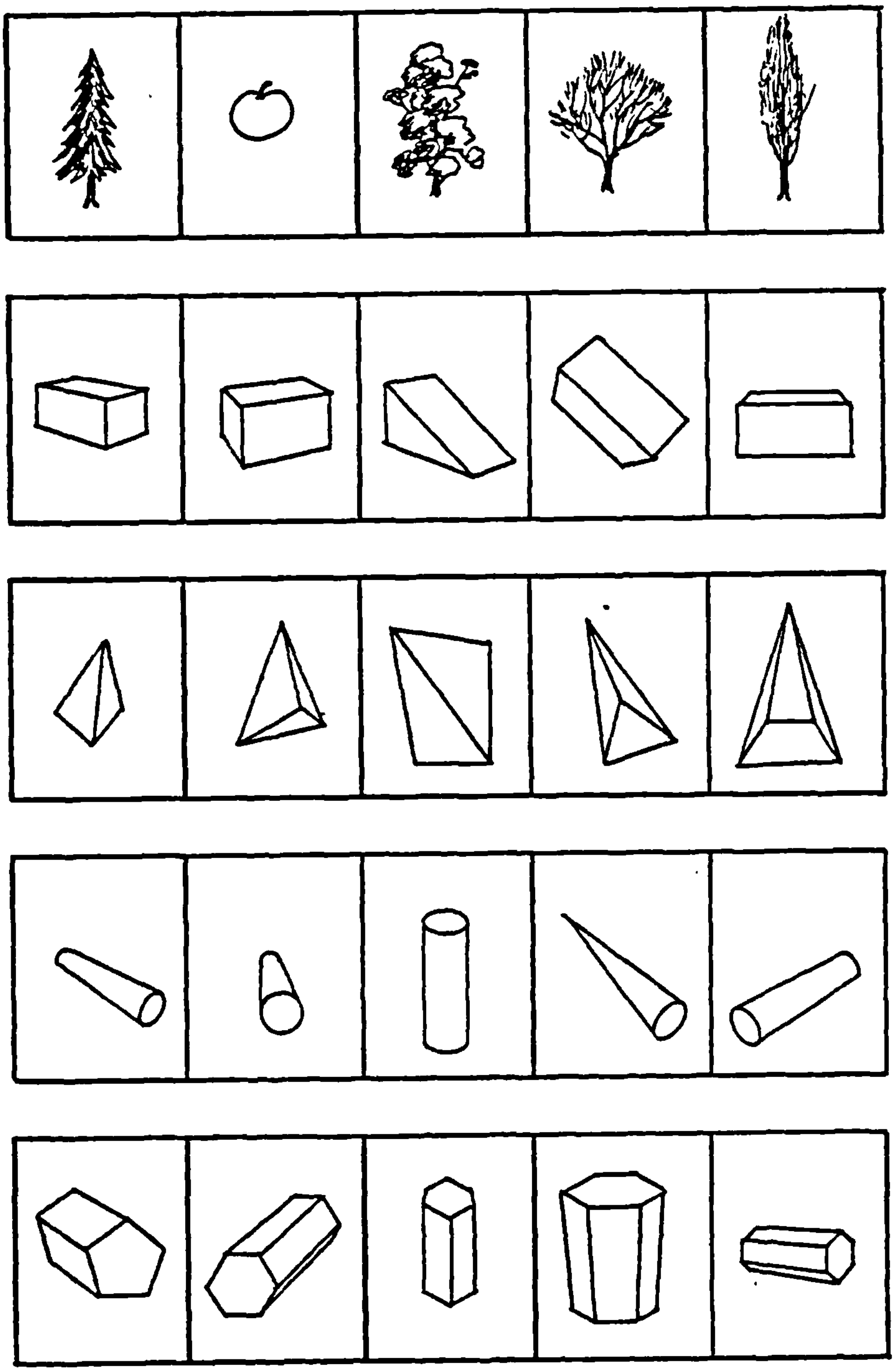
Most of you have crossed the apple - why?

Because all the rest are trees - good - now the others are slightly different.

"In row labelled B all the drawings are of the same object except one which is of a different object, put a cross on the odd one out."

"Now do the same for C, for D and for E."

DISCUSSION Ask the children why they crossed the one that they did. What was specially different about the odd one?



A

B

C

D

E

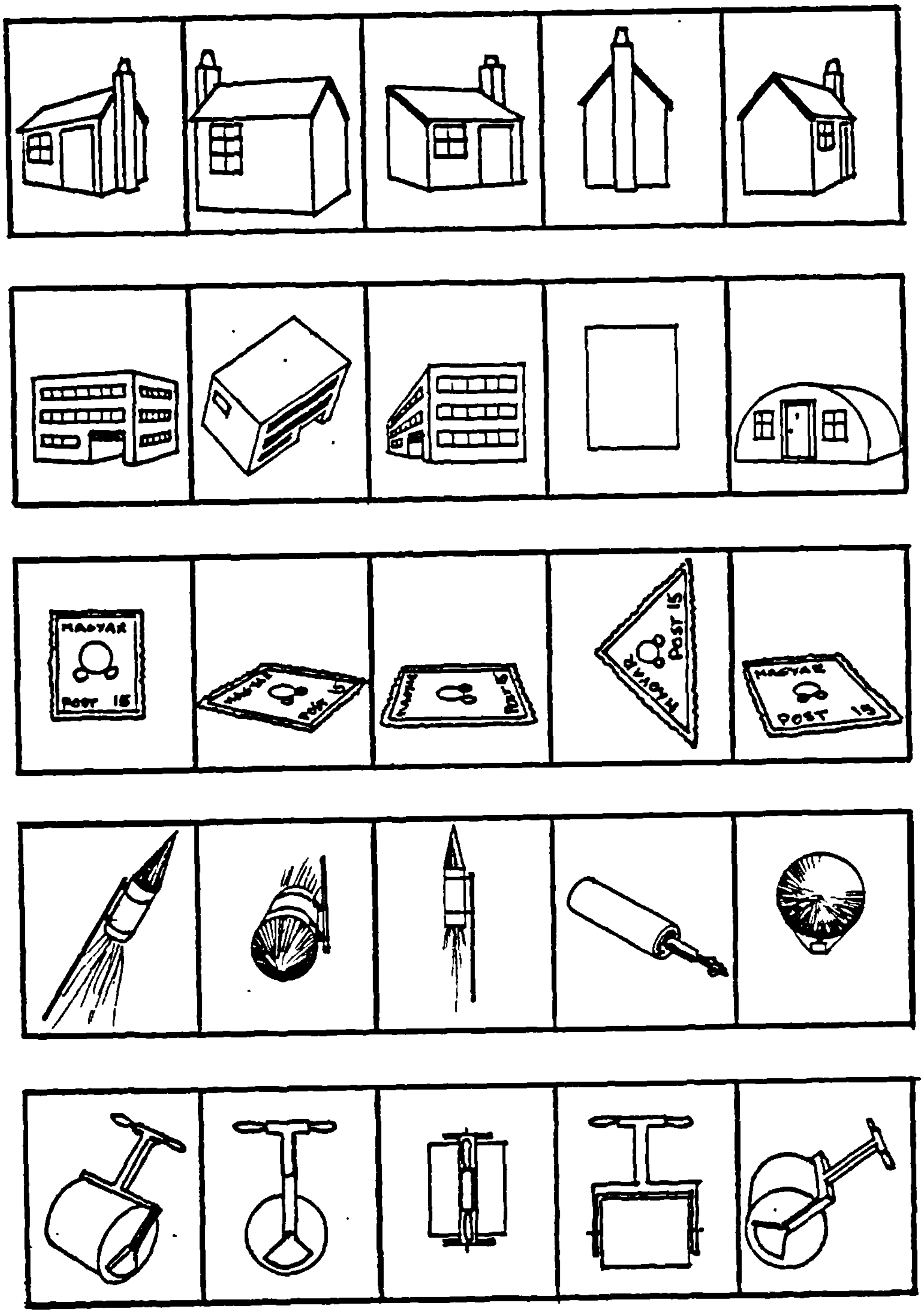
C4

MORE ODD ONES OUT

Multichoice questions. (Projective Viewing)

WHAT TO DO "Put a cross on the picture which is the odd one out. Write a sentence explaining why you chose the one you did."

C4 ctd



C5

SHADOWS FROM A LAMP

Structured drawing, investigation. (Projective Viewing)

AIM Children learn to make shadows of familiar and not so familiar objects. To be able to move a solid figure so that it's shadow fits one already drawn. To be able to recognise an object from its shadow. To make different shadows from one object.

PROCESS Drawing, tracing round a shadow.

MATERIALS Cardboard shapes such as triangles, various shaped quadrilaterals including squares, 5gons and others. circles, stars, and any shape with straight edges, shapes with straight and curved edges, all curved edges. Everyday solids such as mugs, boxes, shoes, satchels, various sized packets, straw models, wire models, solid shapes made up by the children. Bluetack or similar.

WHAT TO DO Make a lamp shine on a plain plane wall. Now make shadows of various objects - what is the shadow of a shoe? Of a book? Of you? See what sort of shadow you can make. Use bluetack to fix a sheet of (news)paper to the wall. Use a felt tip pen to trace some of the interesting and unusual shadows you get. Do not forget to label the shadows. "This is William sitting down." "This is a teapot, this is a star shape." Trace two or three different shadows from the same object. Get some shadows and ask someone to say what the object is. You can do this by making them face the wall and hold the object up to the lamp behind their backs.

DISCUSSION Class discussion could be introduced. Why do you think this shadow is of a mug? Why can't it be of a kettle?

MINIMUM EXPECTED OF CHILDREN Children become familiar with the shapes that a shadow can take.

C6

MORE SHADOWS FROM A LAMP

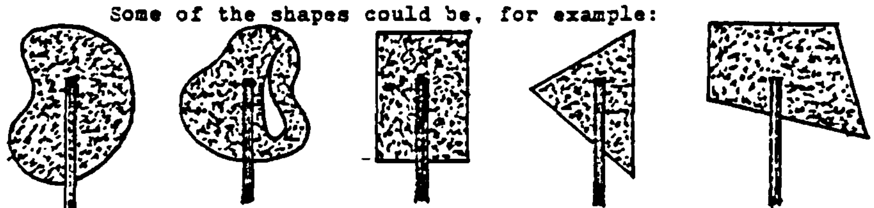
Structured drawing, investigation. (Projective Viewing)

AIM As for 'SHADOWS FROM A LAMP'

PROCESS Drawing, tracing round a shadow.

MATERIALS Two-dimensional cardboard shapes with handles so that the children's hands do not obscure the shape. Some three-dimensional shapes most easily made from wire. Before you give the shapes to the children use the shapes to obtain shadow shapes on a large piece of paper. These shapes should be made by tracing round actual shadows from the cards.

Some of the shapes could be, for example:



and the shadows marked on the sheet of paper could be:

WHAT TO DO Trace round different shadows of these cardboard shapes. Try to make the shadow as thin as you can. Find out which shape makes which of the shadows drawn on the paper.

YET MORE SHADOWS

Drawing, investigation. (Projective Viewing)

WHAT TO DO

Use some interesting flat and three-dimensional shapes such as a cube of wire, a tetrahedron, a figure eight, a 'wiggle' like 

Can a curved object have a straight shadow?

Can the shadow of a figure eight look like a circle?

Make some shapes of your own.

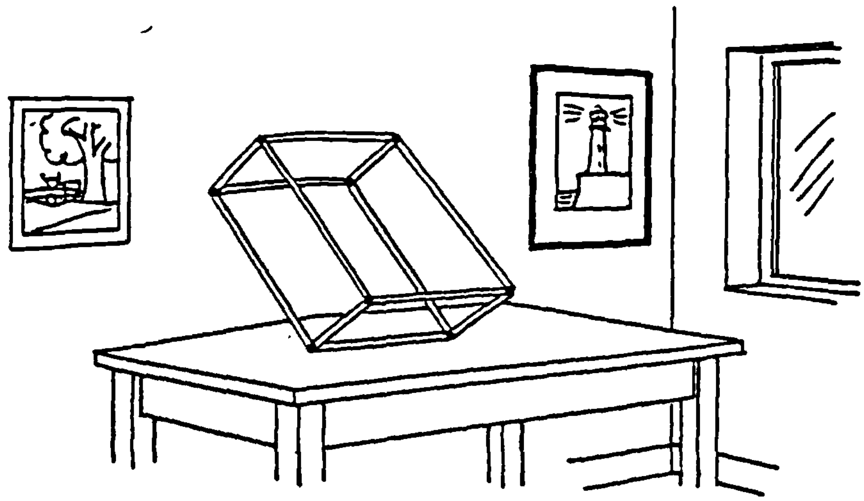
SKELETON DRAWINGS

Structured Drawing. (Projective Viewing).

AIM To involve children in considering relationships between edges and faces of a solid in skeleton form and in its two dimensional representations.

PROCESS Drawing (with ruler if necessary).

WHAT TO DO Place the model of a brick made with milk straws or wire like this:



Draw the model as it looks to you from where you are sitting. Then move to a different place and draw it again.

Remember what you saw when you made shadows from the lamp.

MINIMUM EXPECTED OF CHILDREN An attempt to make their drawings "look right". An ability to discuss how their drawings are not quite right.

ENRICHMENT Make up some other shapes with straws or wire and draw these.

You might choose a tetrahedron, a tent shape or one you make up for yourself.

C9

DRAWING THROUGH A WINDOW

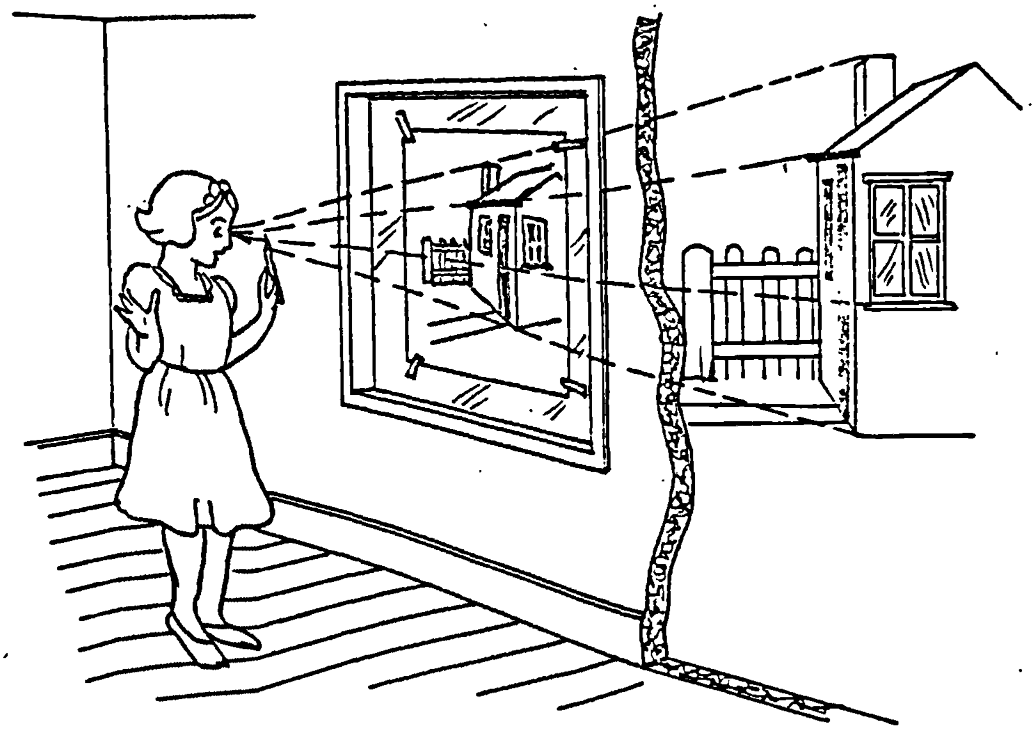
Structured drawing, investigation (Projective viewing)

AIM Experience for children in mapping a three dimensional view onto a plane.

PROCESS Drawing, painting.

MATERIALS Transparent sheet of paper ("cling"), window with a 'good' view (preferably a building and a play area or similar). Felt tip, crayons, paint as appropriate.

WHAT TO DO Fix a piece of "cling" or transparent paper to a window. Try to keep your head still. Then draw on the paper what you see outside. When you have finished you should have a picture of the view from the window on your sheet of paper. Your teacher may let you draw straight on the window pane, but be careful not to press too hard.



C10

WHICH IS THE TALLEST BUILDING?

Multichoice, (Projective Viewing)

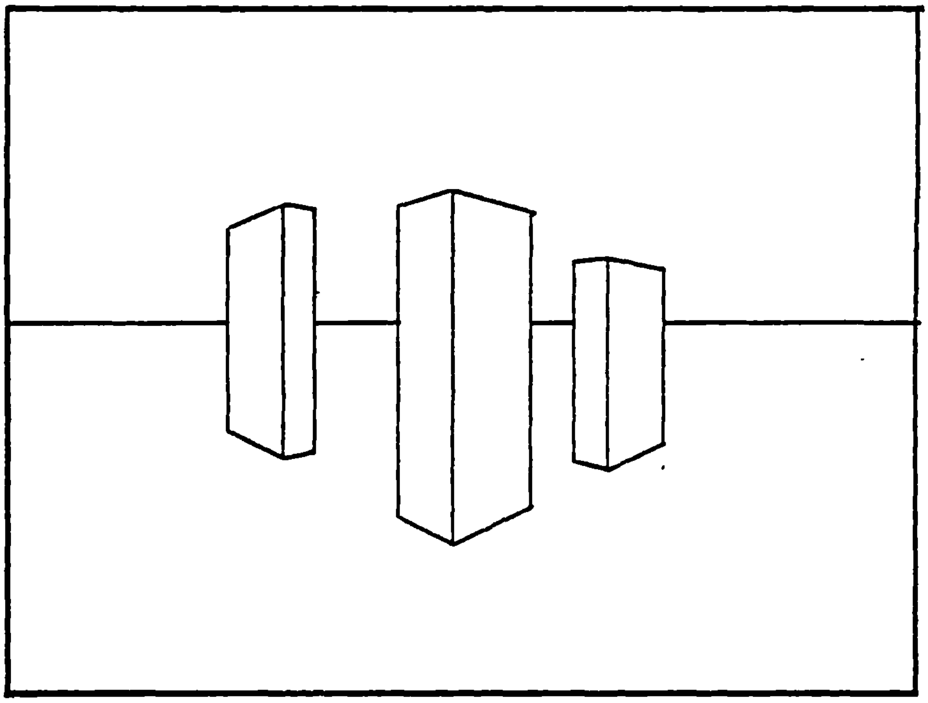
AIM Children learn to interpret a line drawing perspective view. They learn to place objects in order of height, using receding lines as cues.

PROCESS Selection

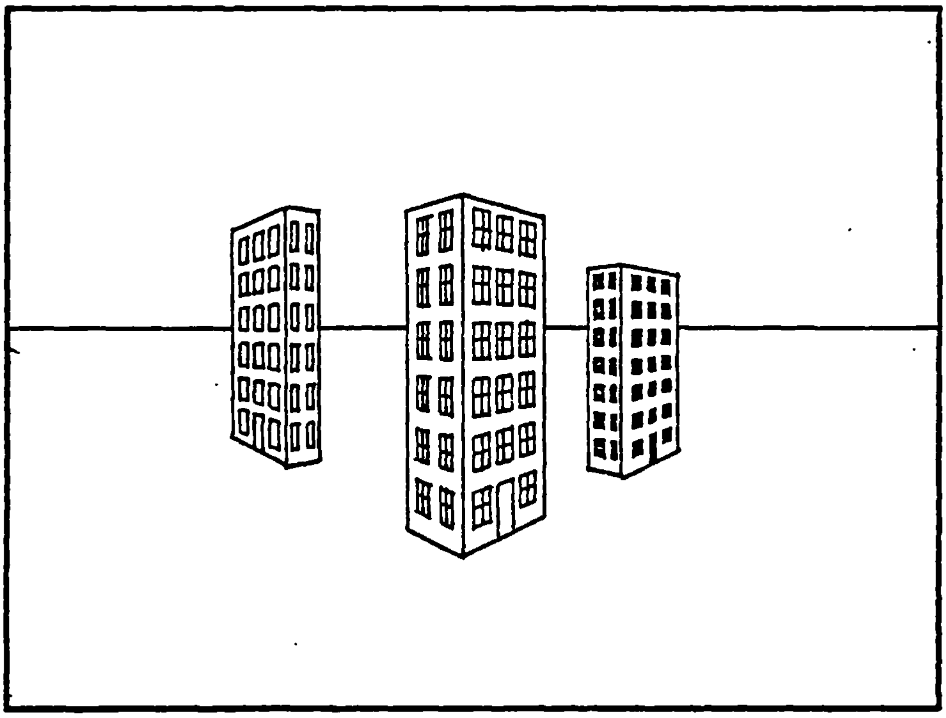
WHAT TO DO Here are three buildings on a flat countryside. Decide which LOOKS the tallest - put a t on this building. Decide which LOOKS the shortest - put an s on this building.

DISCUSSION "Who thinks they have the right answer? "Why do you say the tallest building is......" etc.

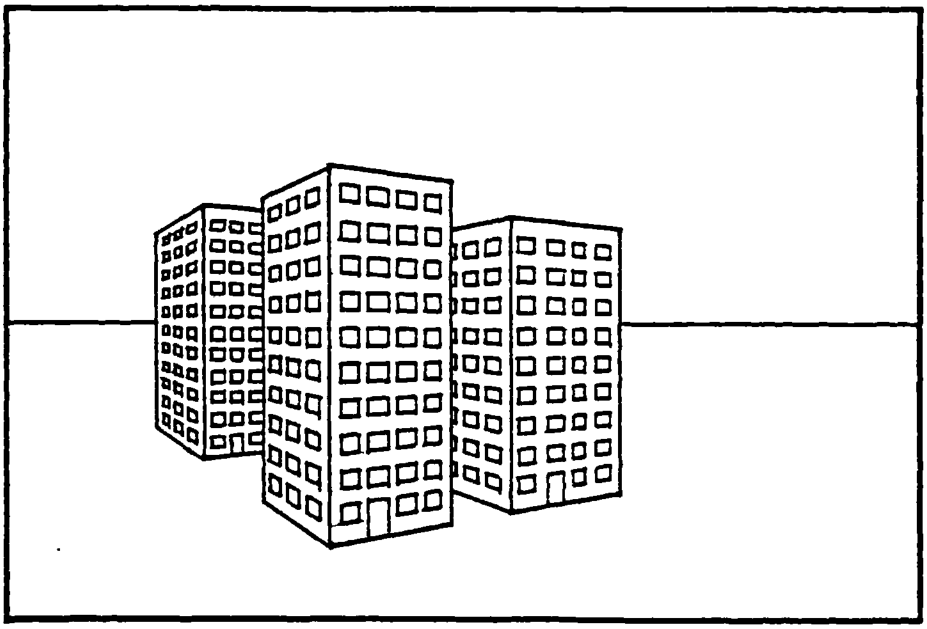
C10 ctd



C10 ctd



C10 ctd



C11

WHICH IS THE TALLEST?

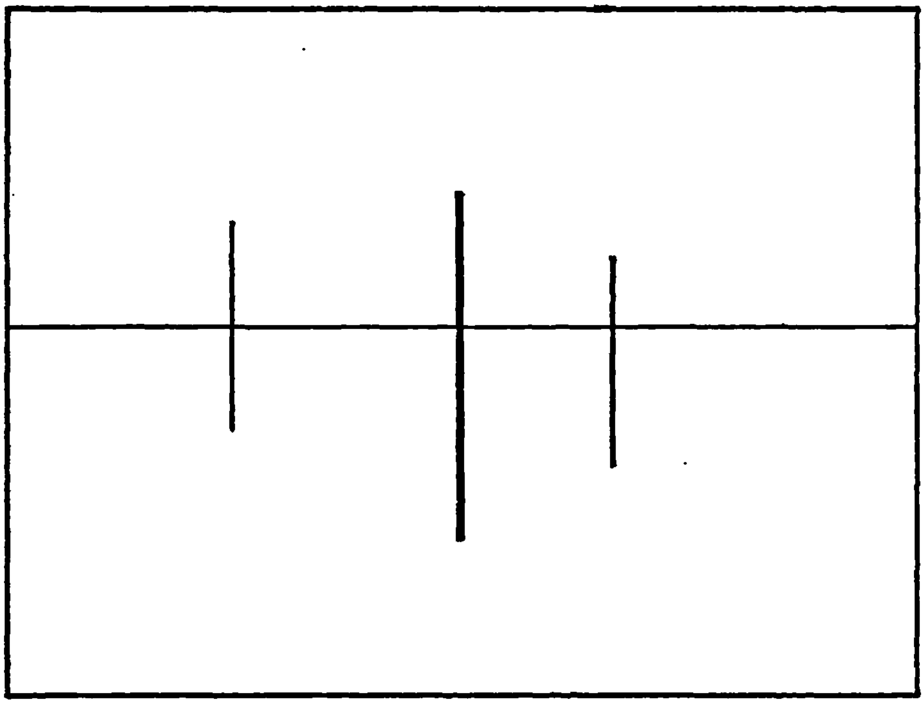
Multichoice. (Projective Viewing)

WHAT TO DO Here are three trees by the side of a lake. Can you see the man in the canoe? Decide which tree is the tallest, which the shortest.

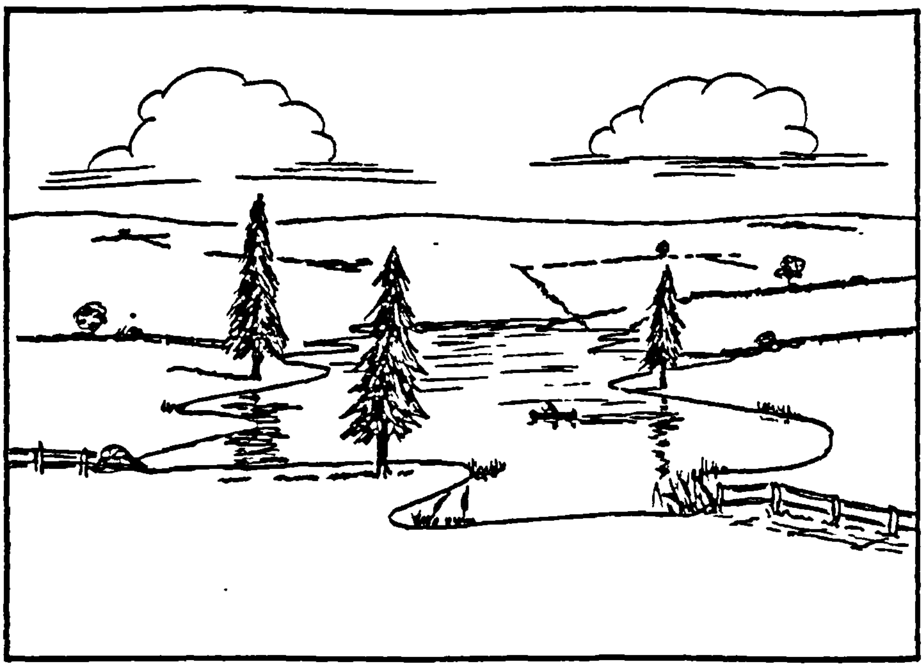
Do the same for the three poles on a flat field.

In the next drawing some extra lines have been added. Does this make you change your mind?

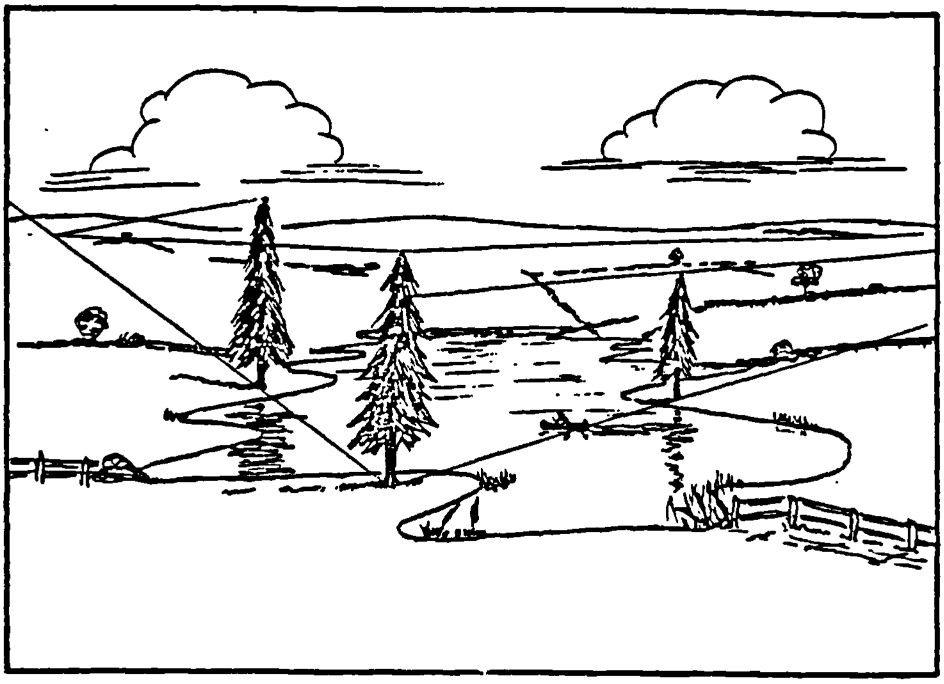
C11 ctd



C11 ctd



C11 ctd



VIEWING THROUGH A PAPER FRAME

Structured drawing, investigation (Projective Viewing)

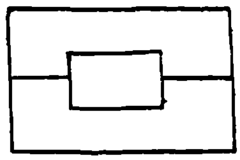
AIM To give children experience of representing a view or an object on paper.

PROCESS Drawing, painting.

MATERIALS Stiff paper or card, scissors, felt tip, crayon or paint as appropriate.

WHAT TO DO Use a piece of card or stiff paper about 15 cm by 10 cm. Fold this in two

Cut out a rectangular piece about 6 cm by 2 cm and open out to have a frame 6 cm by 4 cm in the middle



Use the paper frame to look at things in the classroom. Draw on your paper exactly what you can see in the hole in your card. It might help to fix the card so that it does not move when you are drawing your views. Try drawing part of your desk or a part of a wall or an object you particularly like drawing.

LEARNING That drawing to make things "look right" needs careful observation.

DISPLAY of some of the drawings.

DISCUSSION Ask the children what they found out. Did they notice anything new? Lead into parallel lines appearing to converge. Make links with other exercises such as shadows and drawings through windows.

ENRICHMENT Some children may be asked to write out their drawings. Others may be asked to draw the same object from a variety of views.

D1

WHICH ARE VIEWS FROM ABOVE?

Multichoice selection, (Affine Representation)

AIM To discover how children consider objects viewed from directly above.

PROCESS Selection.

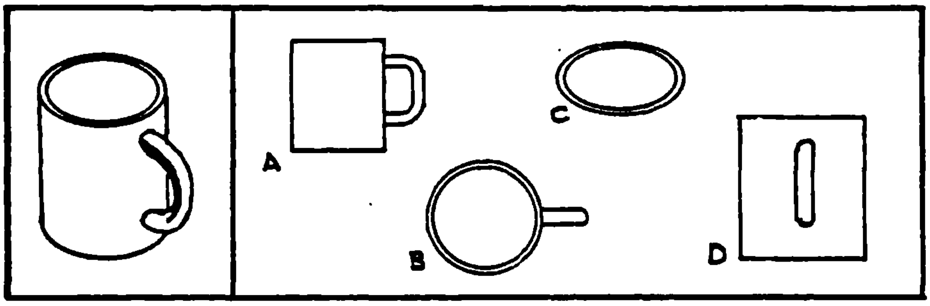
WHAT TO DO

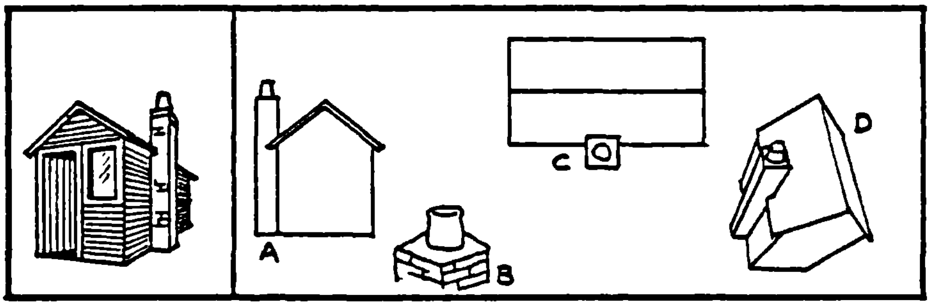
On the left in a box by itself is a drawing of a cup. On the right in the box next to it are four other drawings marked A, B, C, D. Write down which out of A, B, C and D is the view of the object from directly above it.
Do the same for the shed, the wedge and the bottle.

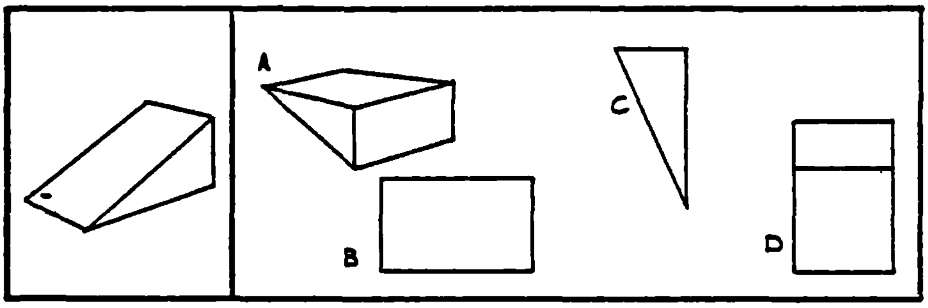
ENRICHMENT

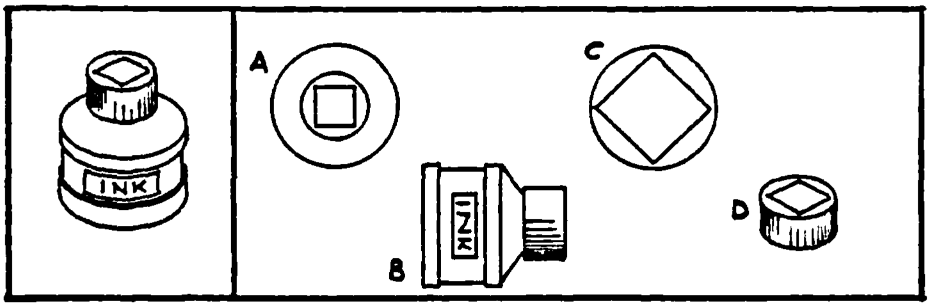
Draw some other shapes from above - you could draw a book, a table lamp, a clock, a church a house or choose one of your own.

D1 ctd









WHICH DRAWING LOOKS MOST LIKE?

Multichoice selection, discussion (Affine Representation)

AIM To discover how children select realistic views from a selection.

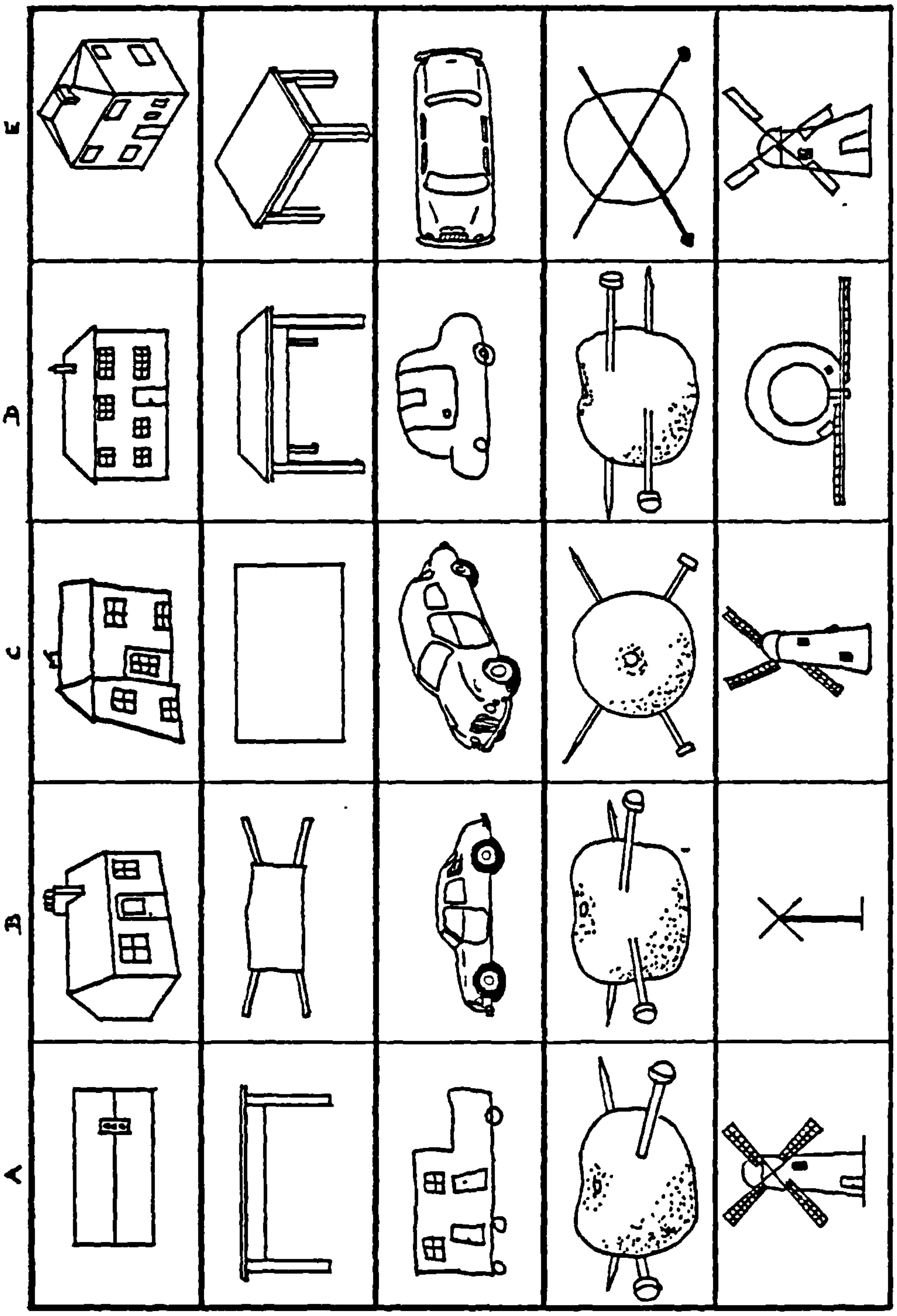
PROCESS Selection.

WHAT TO DO

"Here are five drawings of a house, marked A, B, C, D and E. Decide which one looks most like a house. Write down which one you think A, B, C, D or E. Here are five drawings of a table, car, two knitting needles through an orange, windmill marked A, B, C, D and E. Again decide which one looks most like, A, B, C, D or E and write down your answer.

DISCUSSION Ask children which selections they have made and why they have made them. Does anyone have an unusual choice and why? What is special about the choices that were made?

D2 ctd



D3

VIEWS FROM ABOVE (PLANS)

Multichoice selections, (Affine Representations)

AIM To enable children to distinguish between projective drawings and a plan - by selecting a plan from a variety of projective drawings. Children should learn to draw plans from projective views and from solid objects.

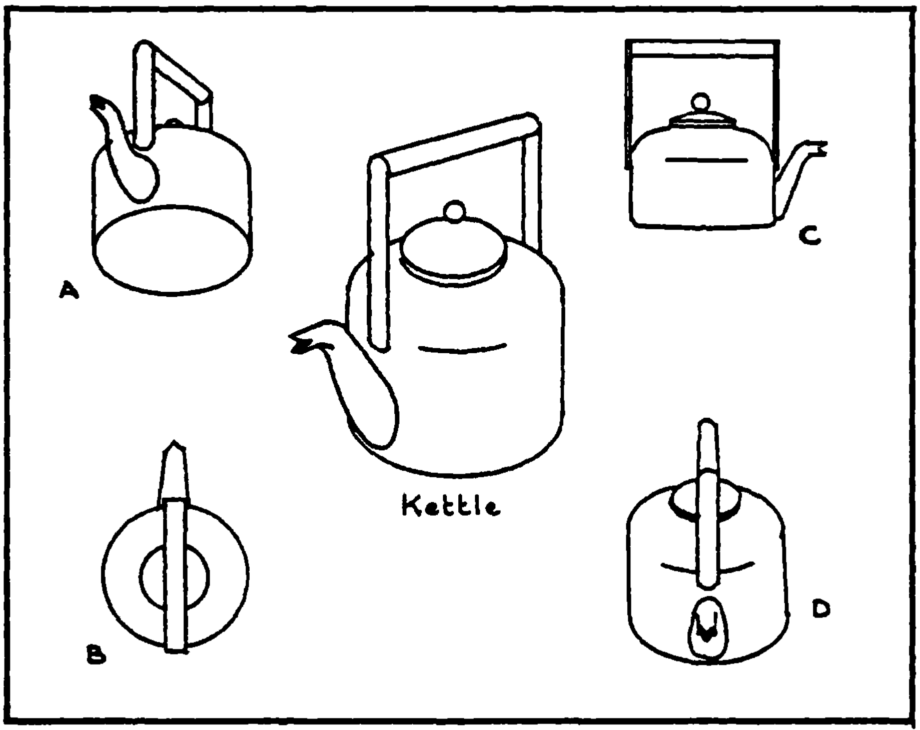
PROCESS Selections and drawing.

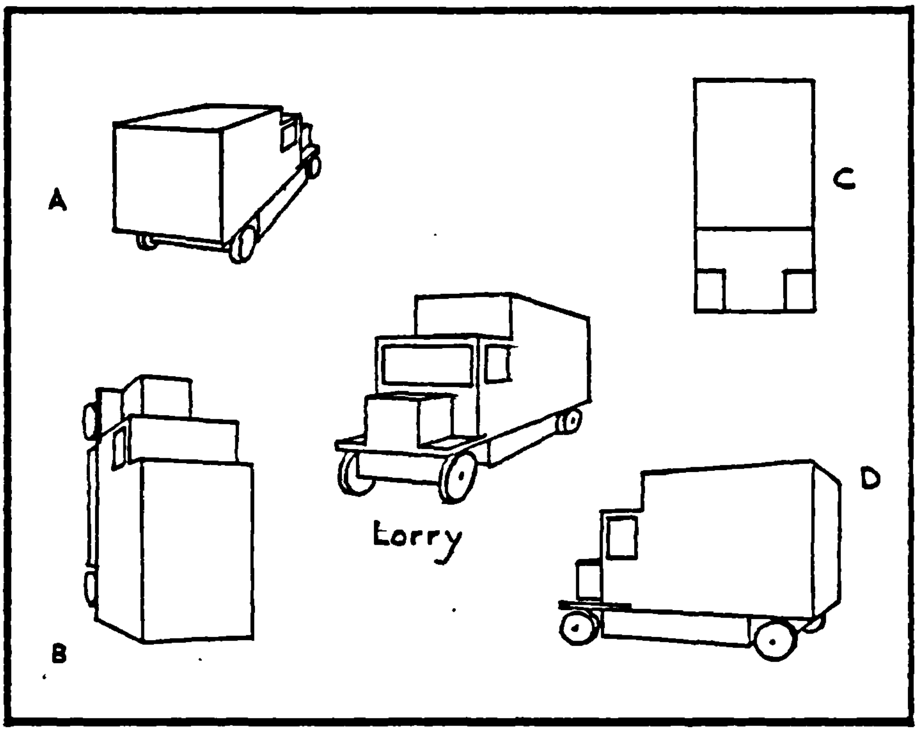
WHAT TO DO

Here are five drawings of a kettle. One of them is the view from above. Write down which letter out of A, B, C, and D. is the one by the view from above. The view from above is called the plan.

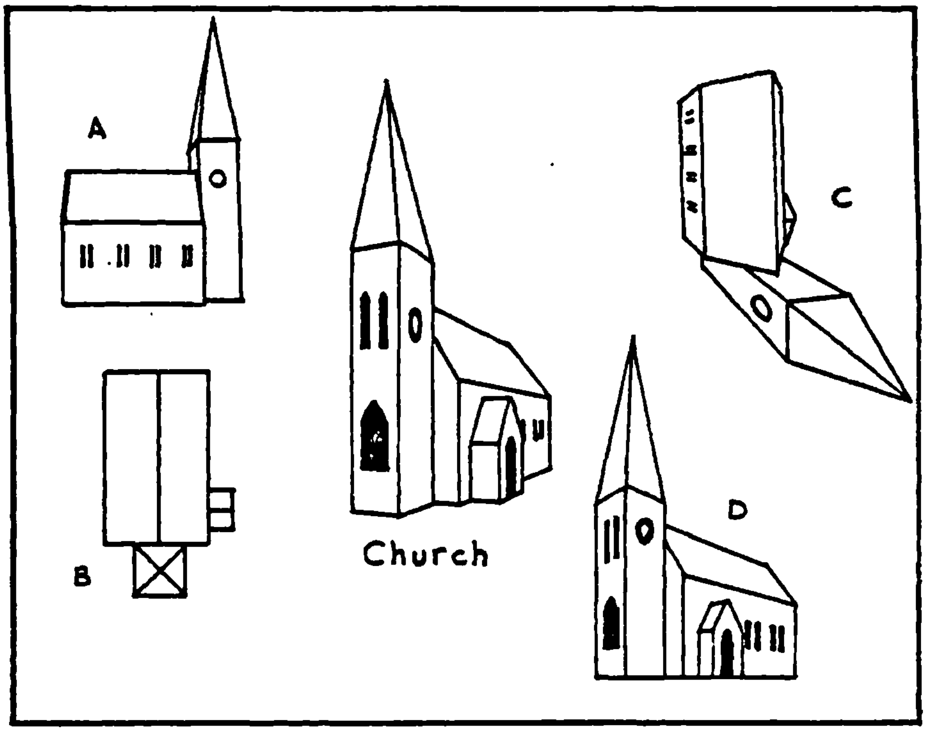
Do the same for the lorry, the church, and the table.

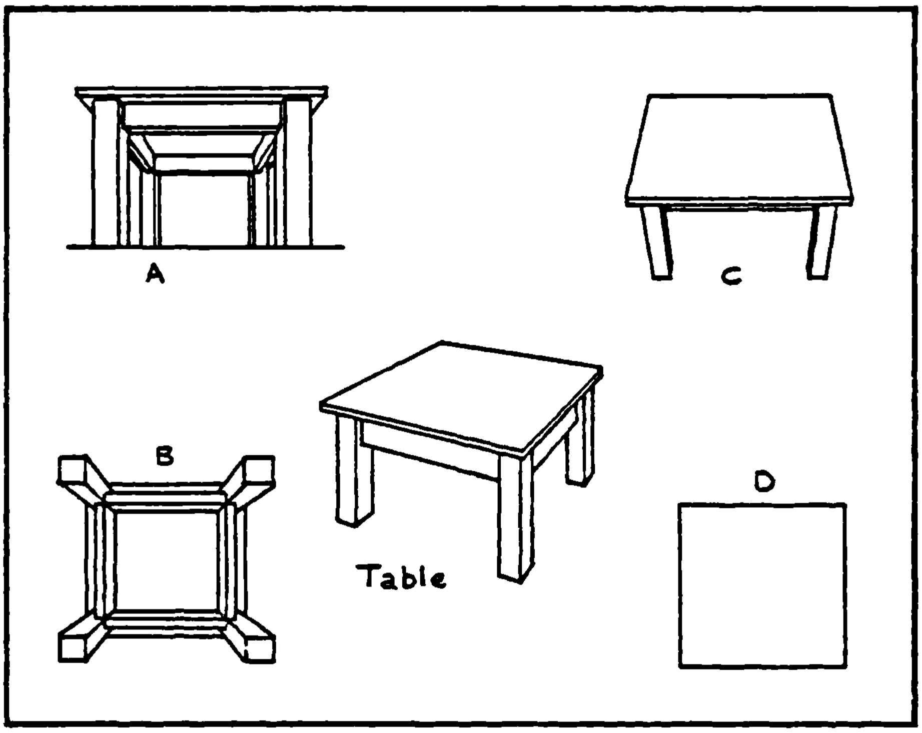
D3 ctd





D3 ctd





D4

DRAWING PLANS

Structured drawing. (Affine Representation)

AIM Children experience drawing plans.

PROCESS Drawing, (painting)

MATERIALS Pencil. (paints)

WHAT TO DO

Draw a picture of a cup and saucer as it would look if you were looking at it from directly above. Draw a plan of a cup and saucer. Do the same for a television set and a table - separately or one on top of the other.

ENRICHMENT Draw a plan of your bedroom or another room in your home.

Draw it (or paint it) as it would look from above. The drawing (or painting) should be a plan.

D5

WHICH IS THE PLAN?

Multichoice selection, discussion (Affine Representation)

AIM To give children further experience in selecting plans.

PROCESS Selection.

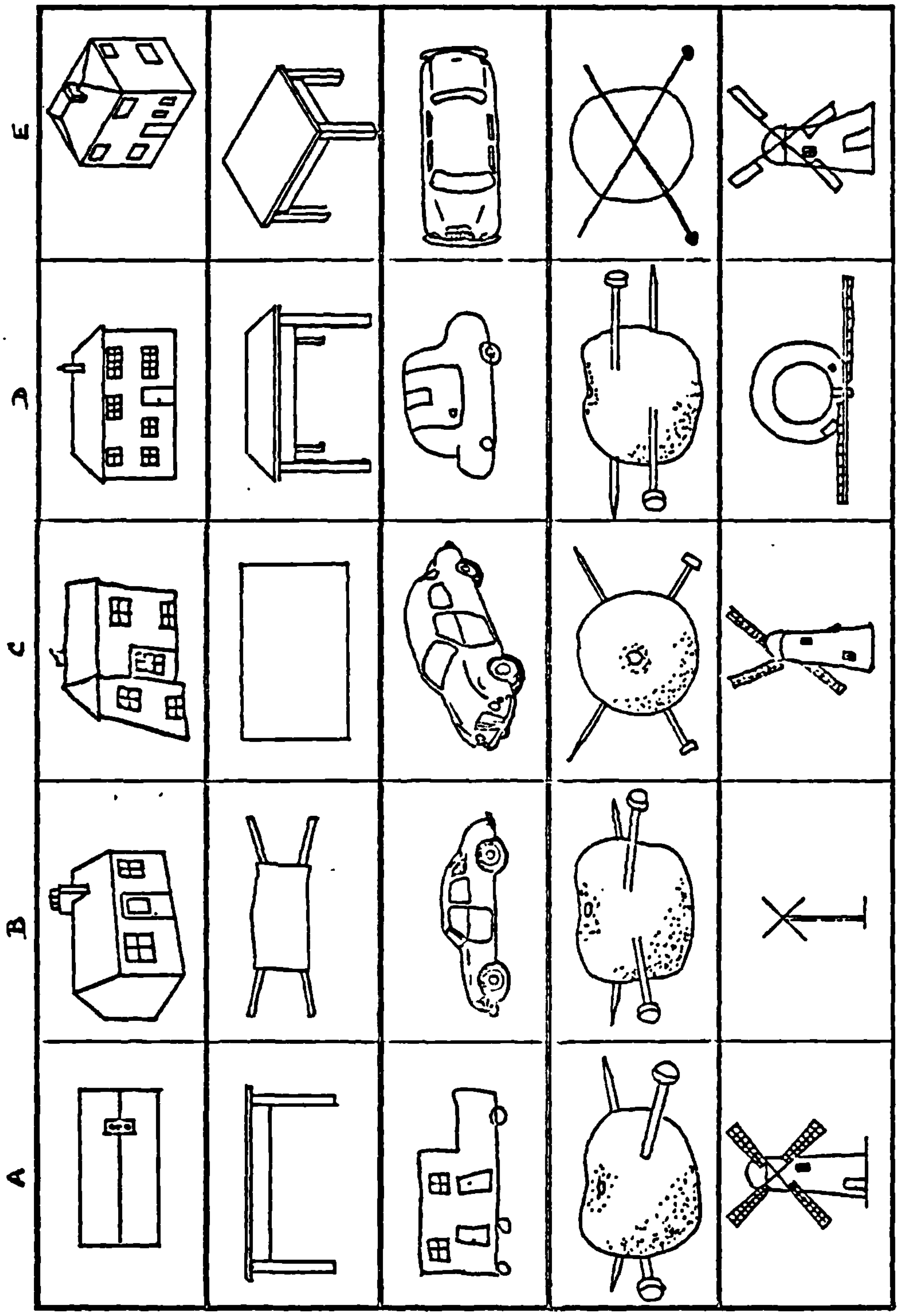
WHAT TO DO

Here are the five drawings of a house, marked A, B, C, D and E which you have met before.

This time decide which drawing is the plan (the view from above). Write down which one you think.

Do the same for the table, car, two knitting needles through an orange, windmill.

D5 ctd



D6

FRONT VIEWS

Structured drawing (Affine Representation)

AIM To discover how children consider objects viewed from directly in front.

PROCESS Drawing. Painting.

MATERIALS Various objects, blocks, matchboxes, kettle, desk, everyday objects in a classroom.

WHAT TO DO

"What does an object look like from the front? Draw or paint a picture of what the front looks like. Remember you are not supposed to be able to see round the side or over the top.
Draw some other objects such as a cup and saucer, a desk, choose some objects of your own."

D7

WHICH IS THE FRONT VIEW?

Multichoice selection, discussion. (Affine Representation)

AIM To give children experience in selecting front views (elevations)

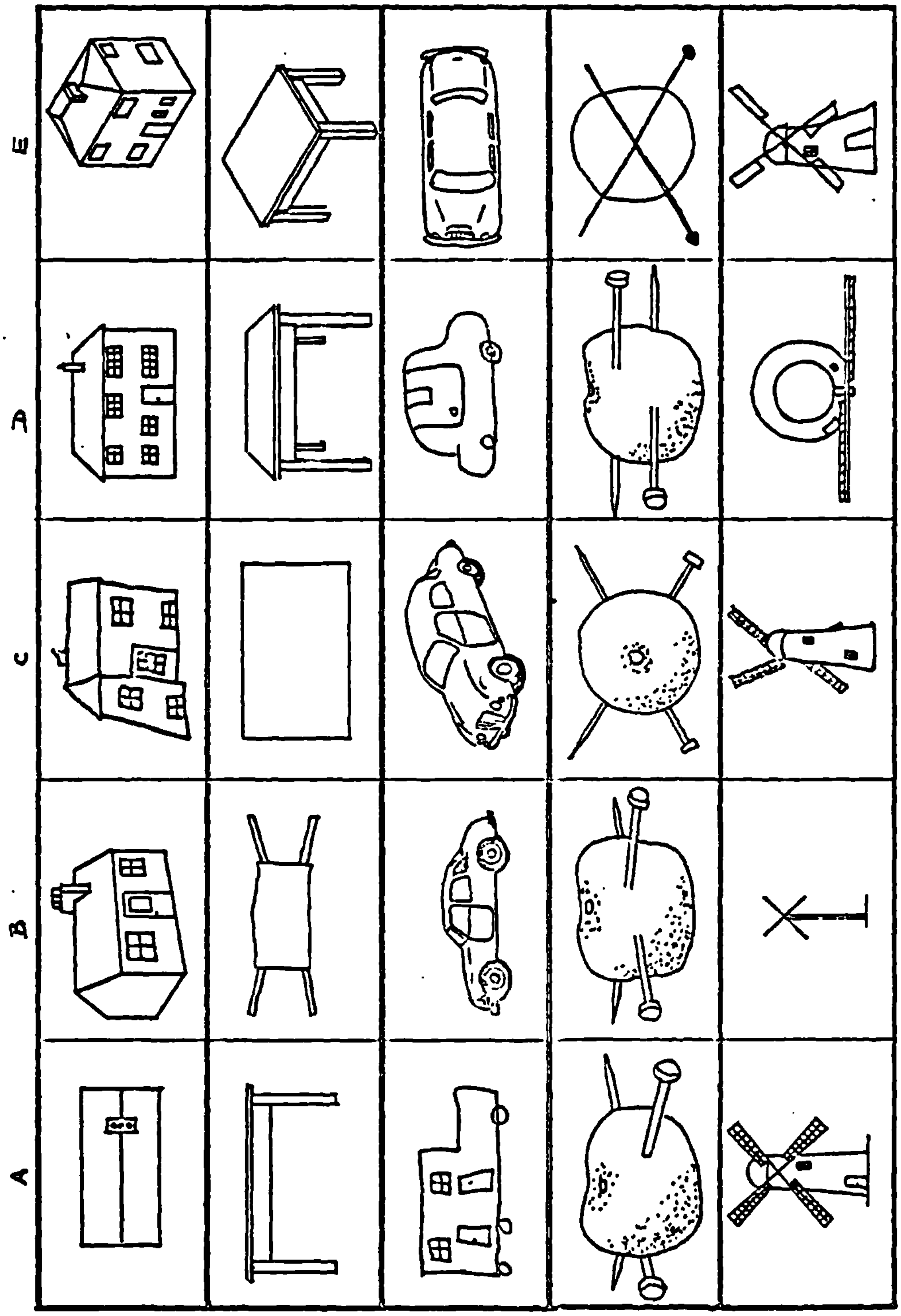
PROCESS Selection.

WHAT TO DO

Once again, here are five drawings of a house marked A, B, C, D and E. This time decide which drawing is the view from in front Write down which one you think. Do the same for the table, car, two knitting needles through an orange, windmill.

DISCUSSION As before, ask children which selections they have made and why.

D7 ctd



SOLIDS FROM DRAWINGS

Interpretation of perspective views, construction (Affine Representation)

AIM Children learn to interpret a perspective drawing by constructing a three dimensional model

PROCESS Construction

MATERIALS Building blocks, cubes, various geometrical solid shapes or structural rods such as Stern, Colour Factor, Cuisenaire or most conveniently, Centicubes.

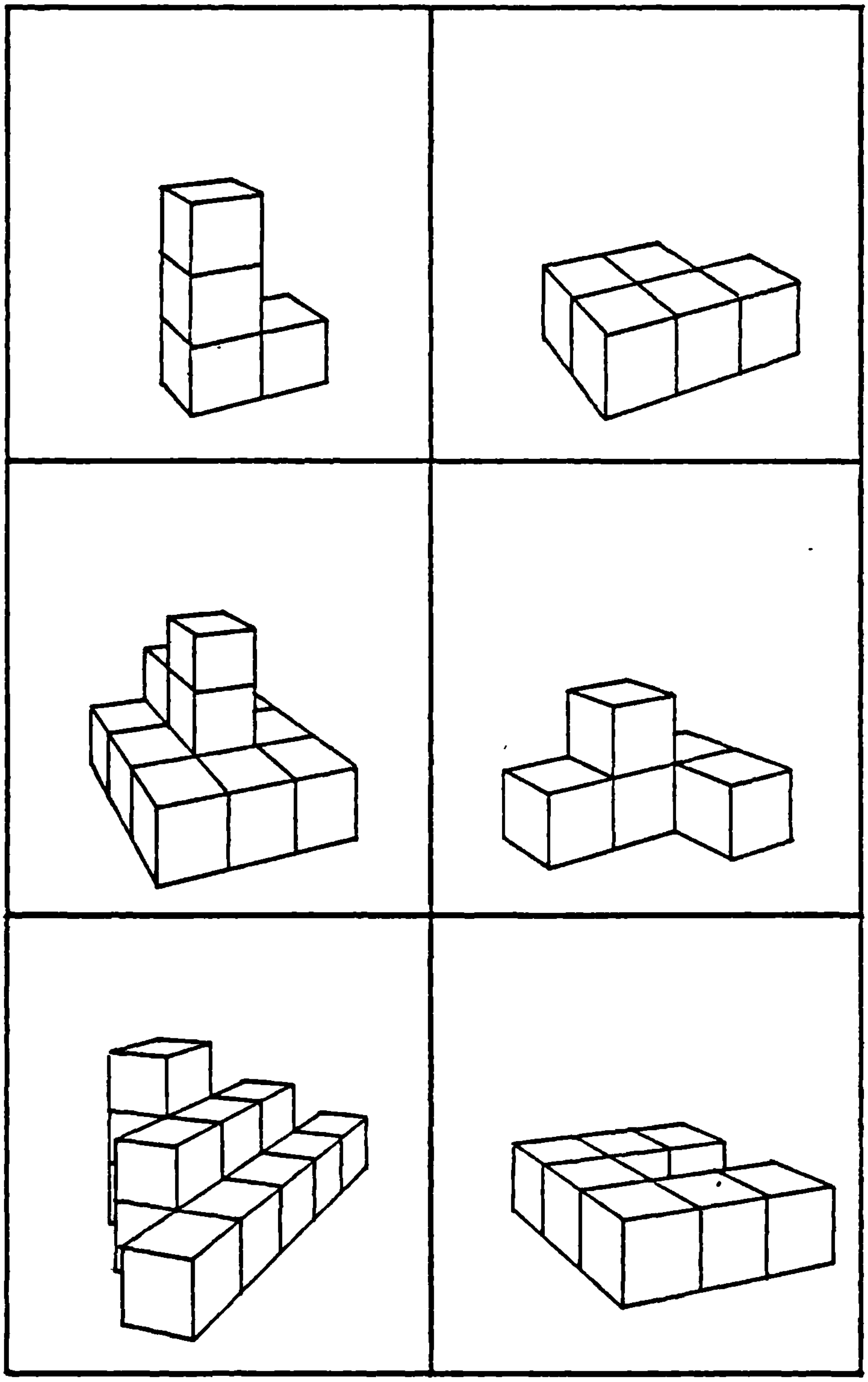
WHAT TO DO

Here is a drawing of a shape Make the shape out of cubes or rods. Make the other shapes as well - there are no cubes hidden behind the others. When you have made a shape, look at it from above and draw the plan. Then look at it from the front and draw the view from the front (front elevation).

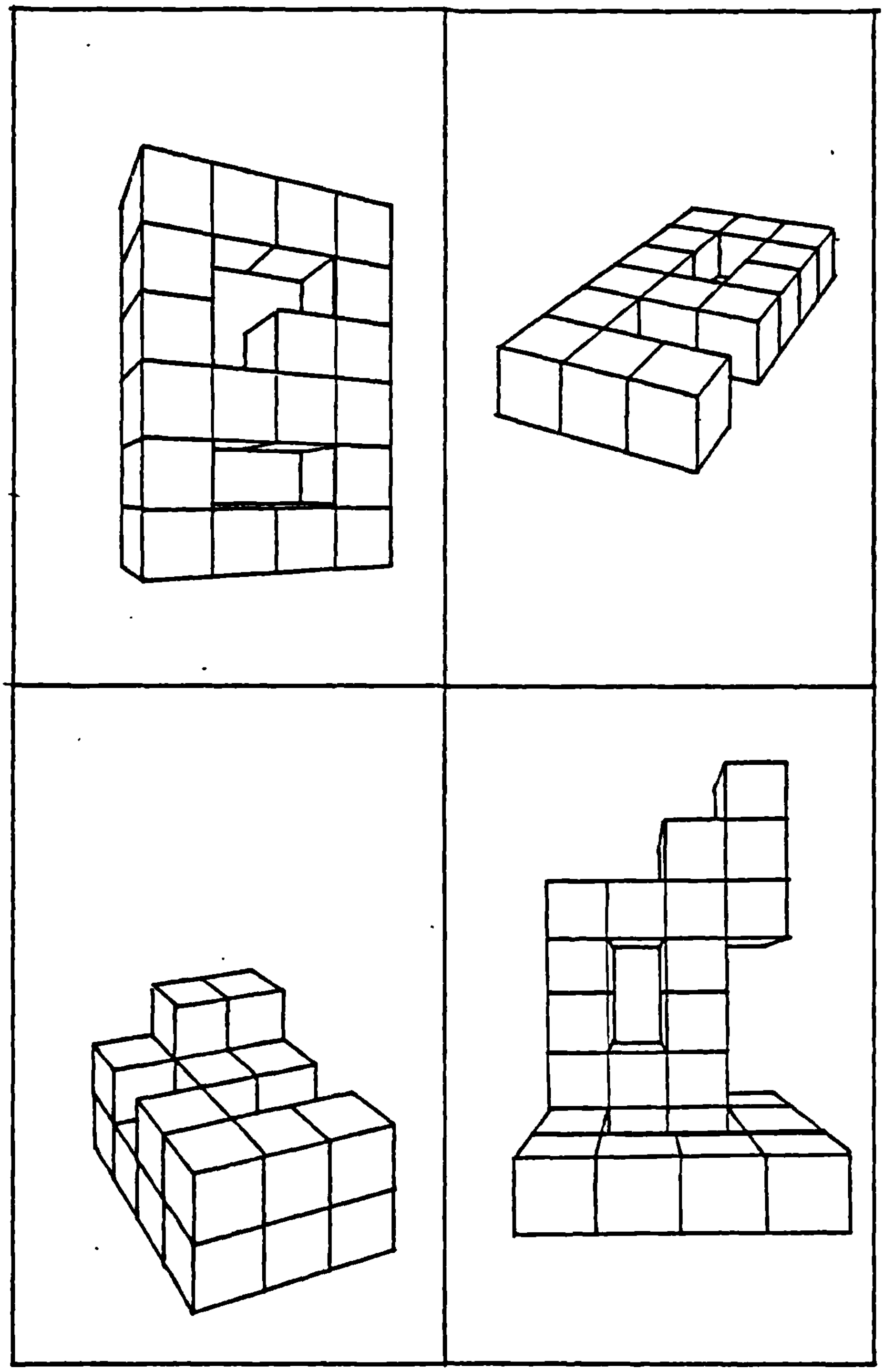
ENRICHMENT

Some children may be encouraged to draw also the side elevations and elevations from behind.

D8 ctd



D8 ctd



MODELS FROM DRAWINGS AND PLANS

Construction. (Affine Representation)

AIM To challenge children with the problem of making a model of an area - school ground - farm buildings - bungalow, from the information available in a projective view and a plan.

PROCESS Construction, painting, making shapes from nets, pasting, glueing, measuring.

MATERIALS Anything to hand, card, cardboard, paper, papier mache, straws, wood and so on.

WHAT TO DO

Make a model of the area shown in the picture. You have a plan to help you as well. Try to make it look as right as you can. You will have to take trouble to make the buildings the right size. Some of you may be able to make some people to put in your model.

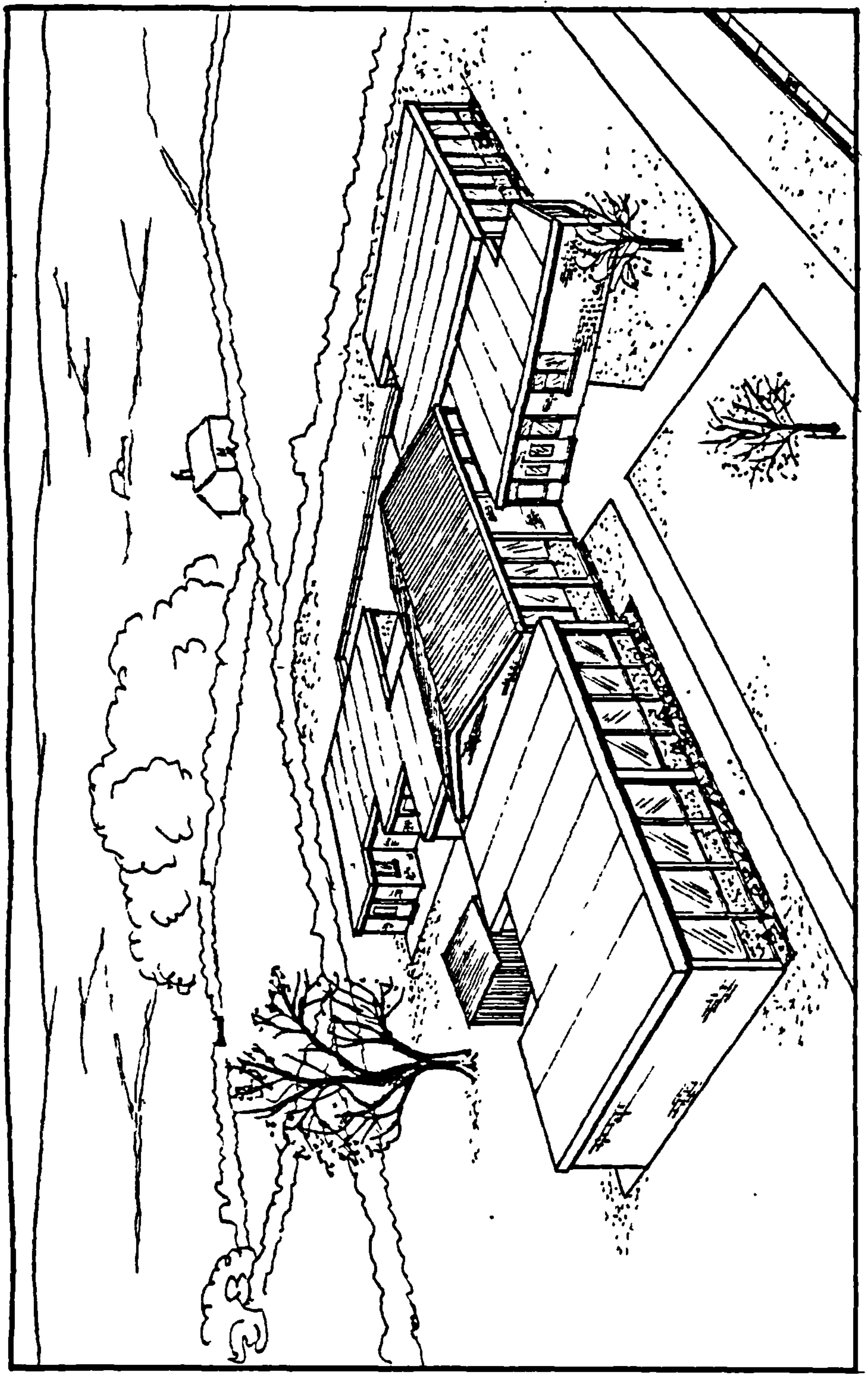
DISCUSSION May be necessary to show what is required. Painting in or use of sticky paper for windows and doors may be discussed. What colour should the road be? Do we paint that before we put the houses - buildings - what have you - on? Which areas should we paint green, or brown, or blue?

DISPLAY The model is its own display.

ENRICHMENT Some children could be asked to do more, perhaps check on an intuitive idea of scale, perhaps embellish the plan or another copy of it, preferably to show the features which can be seen on the picture but not on the plan.

MINIMUM EXPECTED OF CHILDREN That children begin to relate the various parts of the area to each other Drawing to scale is not essential, rely on the child's interpretation of "looking right."

D9 ctd



D9 ctd

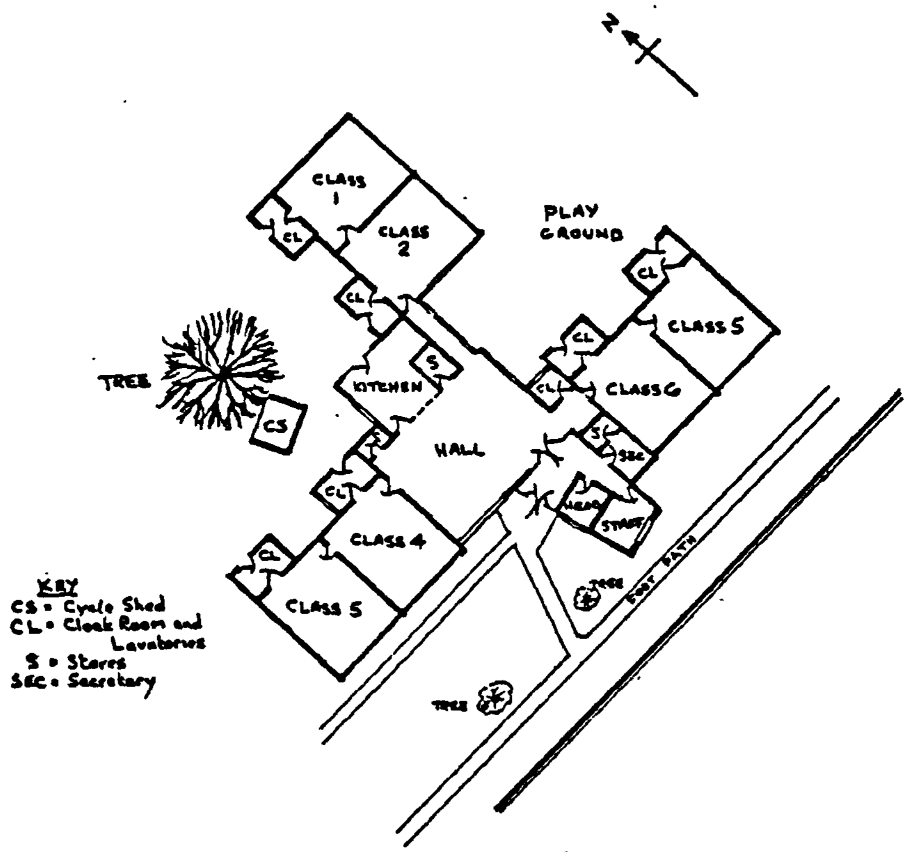
WHAT TO DO NOW

Look at the plan. It is rather bare. It does not show all the things that the picture does. Make some flat shapes to put on the plan to show where the buildings are, what colour they are. You may also use little pictures or symbols to show where things are - like pictures of cars, or people, or traffic signs or tractors or goalposts or whatever.

DISCUSSION Should range over the most appropriate symbolism. How on a plan of an area it makes sense to have a plan of a house rather than a front elevation. But if the concensus of opinion is that a front elevation is "more use" - "easier to see" or similar then it is appropriate to leave the plan in the form that the children consider useful, rather than to impose an artificial adult convention on their experiences. Even Ordnance Survey maps use "icons" such as the windmill sign rather than sumbols for some objects. Undue emphasis on scale and its accuracy will detract from the satisfaction of obtaining a good model by cooperative effort.

LEAST EXPECTED Some ability to use the model, or plan or picture to show paths and routes. Some ability to describe parts of the area verbally, or in writing. Some children, and possibly adults, will find this exercise very difficult. Judgment must be used as to its value.

D9 ctd


N
CLASS 1
CLASS 2
PLAY GROUND
CL
CLASS 5
CLASS 6
S
KITCHEN
TREE
CS
HALL
SEC
HEAD
STAFF
CLASS 4
CLASS 5
FOOT PATH
TREE
TREE
KEY
CS = Cycle Shed
CL = Cloak Room and Lavatories
S = Stores
SEC = Secretary

D10

USING MORE PLANS

Structured writing, discussion (Affine representation)

AIM To give children experience in using a plan topologically, that is, to see connections from one point to another; to trace certain routes with a finger, or consider routes verbally and in writing.

PROCESS Structured writing, discussion.

WHAT TO DO

Here is a plan of a table in a classroom. There are also plans of six chairs. The children's names are written on the chairs.

Copy these sentences and fill in the missing words. The missing word is in brackets at the end. Choose the right one.

1. Paul and ________ sit next to each other (Sue, Bill, John)
2. Bill sits opposite ________ (Sue, Mary, Jane)
3. John sits on Paul's ________ (left, right)

   You should answer 'left' or 'right'.
4. Sue sits on Mary's ________ Be careful here - you may need to turn the drawing round (left, right)
5. ________ sits between Paul and Mary (Bill, Jane, Sue)

Now draw a plan of your school table or desk, and label the chairs or seats.

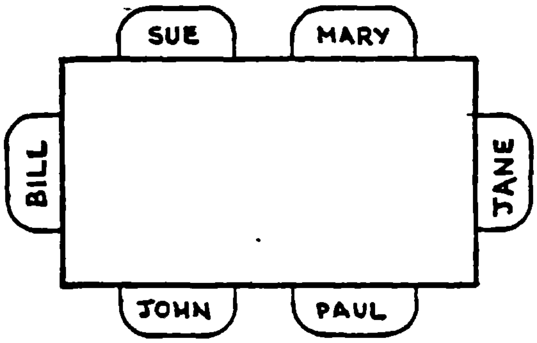
This is a plan of the whole classroom. You should be able to see where Bill's table is and where he sits.

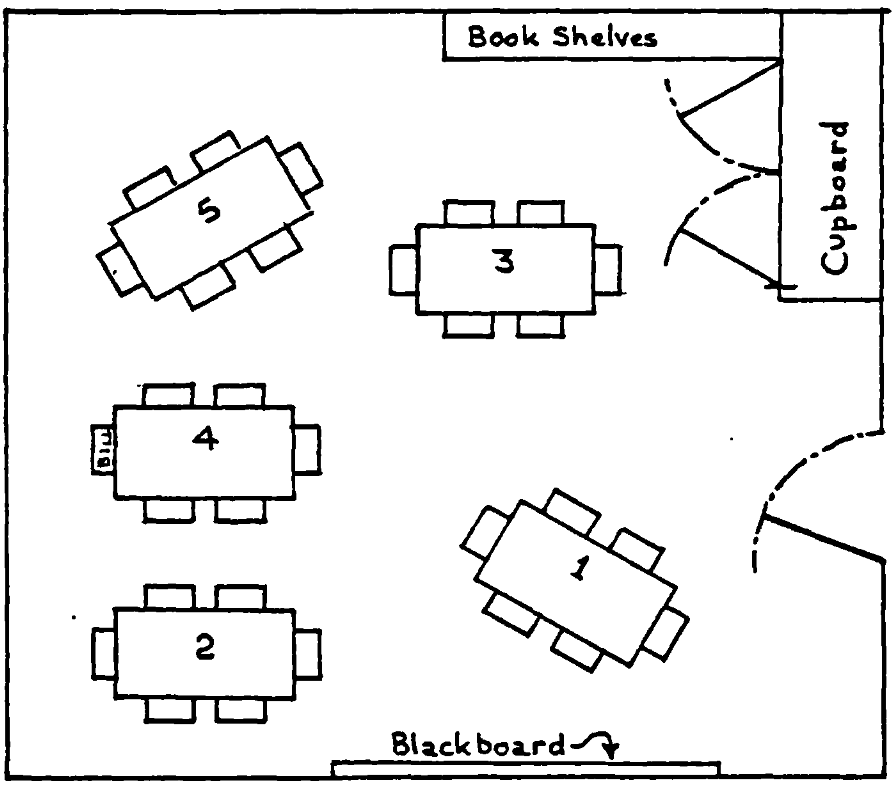
Copy and complete these sentences.

6. Bill's table is number ________ (1,2,3,4,5)
7. This is a ________ of the classroom (drawing, picture, plan)
8. The door is between the ________ and the ________
9. There are ________ chairs in the classroom (28,29,30,31)

Make up some questions of your own.

D10 ctd


SUE
MARY
BILL
JANE
JOHN
PAUL


Book Shelves
Cupboard
5
3
4
BILL
1
2
Blackboard

D10 ctd

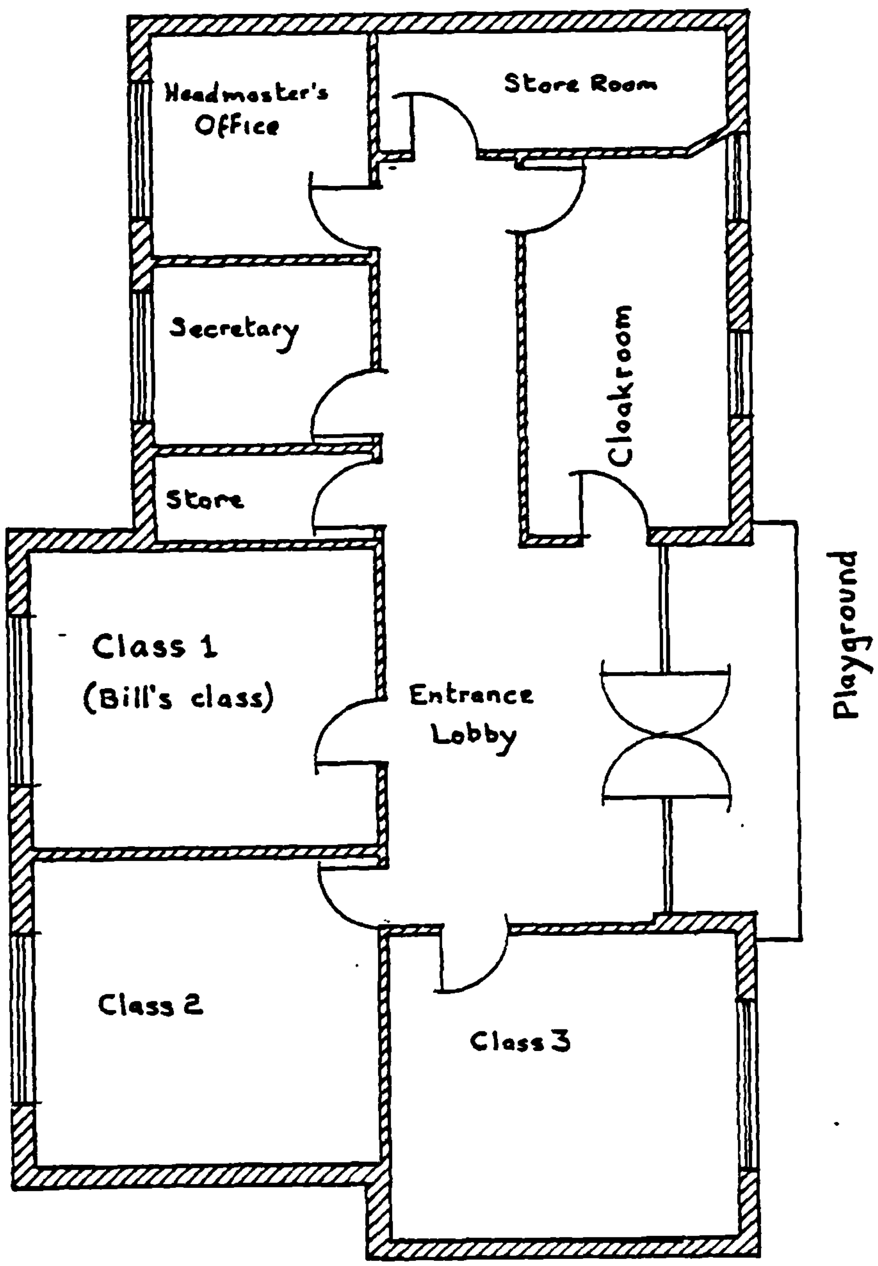
This is a plan of the whole school.

10. Put a small d (d) where the doors should be.

11. How would someone get from Class 2 to the cloakrooms? Write down your answer.

12. Imagine you are in the playground. Draw a picture (front view) or what you think the school would look like from the playground.

D10 ctd


Headmaster's Office
Store Room
Secretary
Cloakroom
Store
Class 1
(Bill's class)
Entrance Lobby
Playground
Class 2
Class 3

D11

WHICH DRAWING LOOKS MOST REAL?

Multichoice selection, (Affine Representation)

AIM To give children experiences in comparing affine representations (in which parallel lines are drawn parallel with projective representations (in which parallel lines converge as they recede to the horizon).

PROCESS Selection, and discussion if necessary.

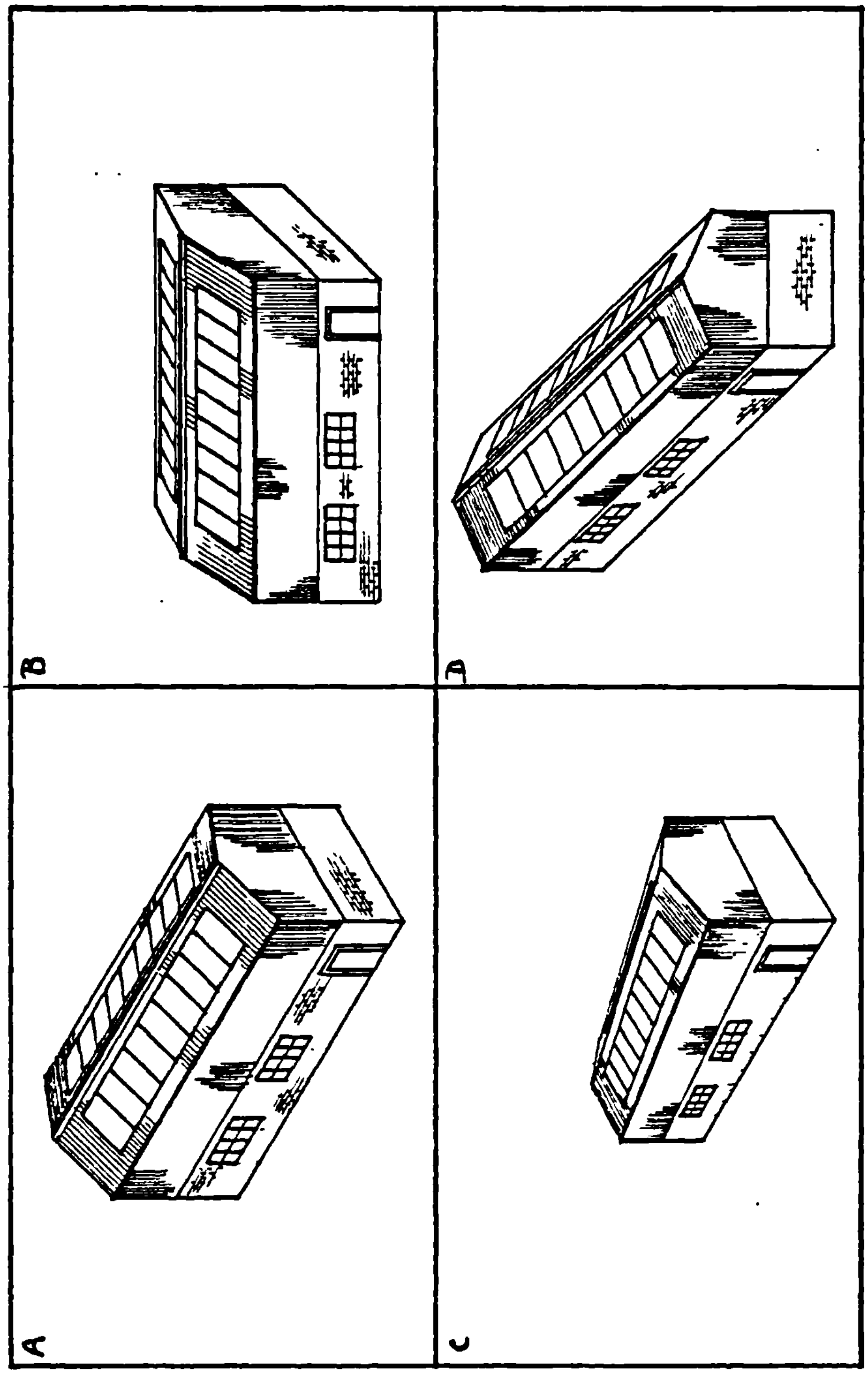
WHAT TO DO

"Here are four drawings of a large factory. They are marked A, B. C and D. In some of them the far end looks too large. Which one has lines receding to the horizon and looks most real?

Write down your answer A, B, C or D. Try to explain why you made your choice.

Write the reason for your choice."

D11 ctd



E1

SEEING THINGS

Selection, discussion. (Visual Perception)

AIM Children experience some examples of visual illusions.

PROCESS Selection, writing.

WHAT TO DO In the first diagram there are three dots. Is the middle dot nearer the dot on the left, the dot on the right, or is it the same distance from each of them?

On the sheet put a cross on "nearer the left", "nearer the right" or "the same."

In the second diagram, which length between the arrow points is longer?

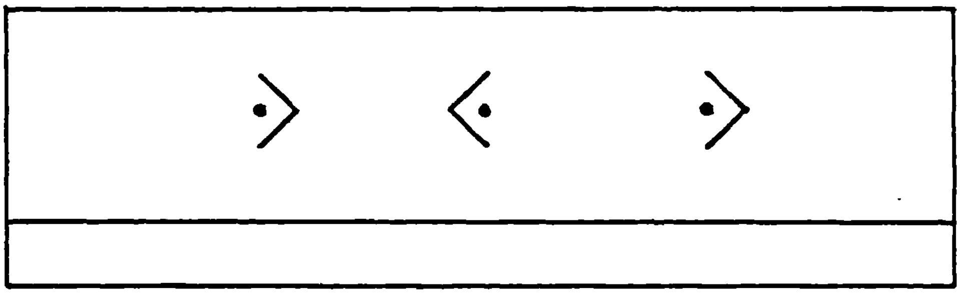
On the sheet put a cross on "a is longer" "b is longer or "the same".

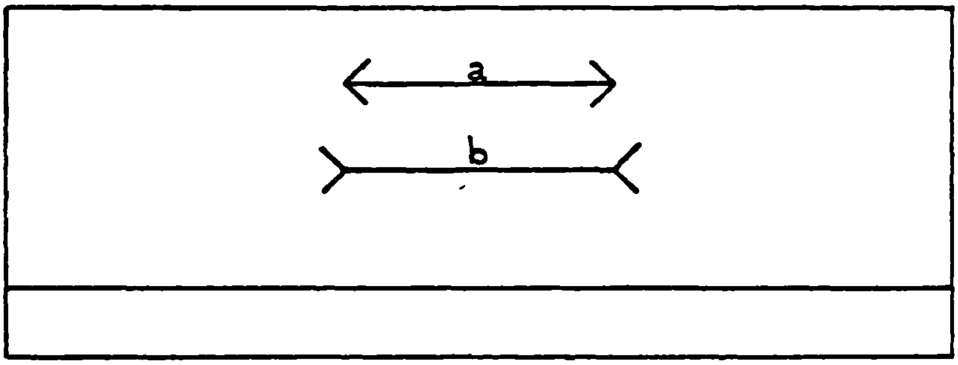
In the third diagram which length is longer? "a is longer", "b is longer", or "the same".

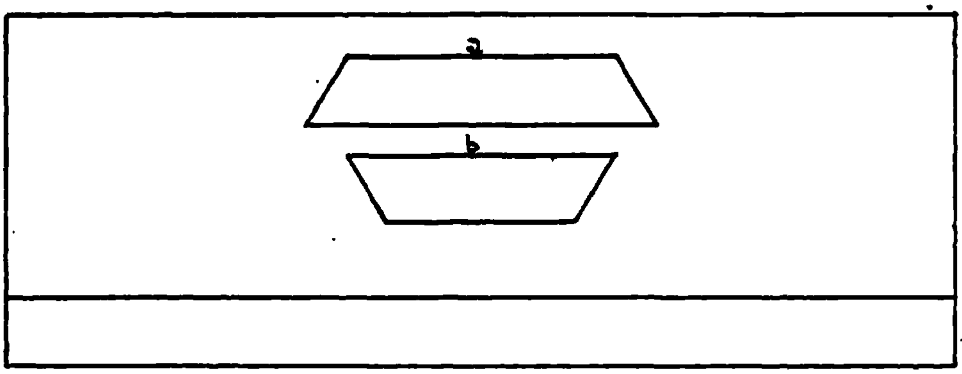
In the last diagram, which is longest out of a, b and c? Write down what you think.

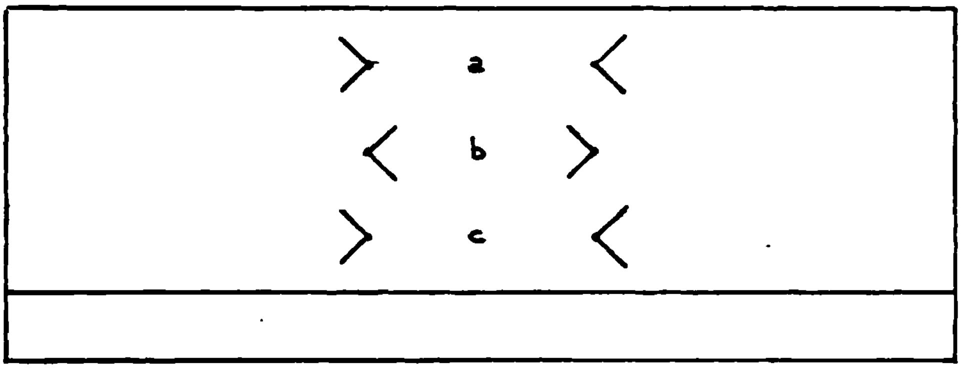
DISCUSSION Could probe the children's ideas.

E1 ctd









LOOKING AT THINGS

Investigation, discussion. (Visual Perception)

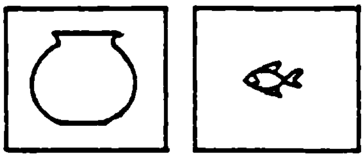
AIM PROCESS as in 'seeing things'.

WHAT TO DO

1. Which upright post looks tallest? Why? guess first and measure them afterwards.
2. Do lines 'a' and 'b' look straight or bent? Are they really bent? How could you tell?
3. Try to count the blocks in this drawing. Why could you have trouble?
4. Are you looking down onto the stairs or up at them from underneath? Turn the drawing slowly. What do you notice?
5. Look at this drawing of a cube. Can you make it change? Which is the face in the front?

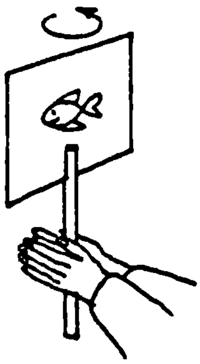
ENRICHMENT

PERSISTENCE OF VISION



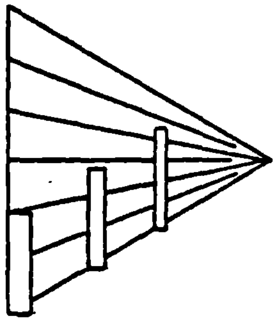
The two sides of the card

Spin the card on the stick and watch the pictures.
What happens?
Does it happen at all speeds?

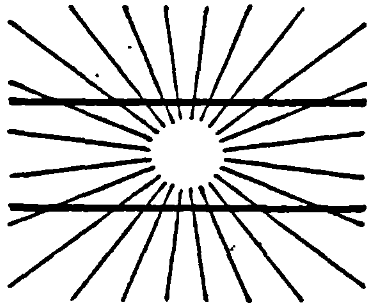


E2 ctd

1



2



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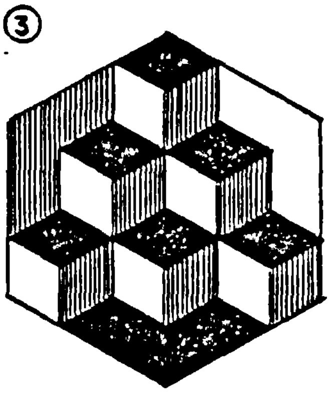

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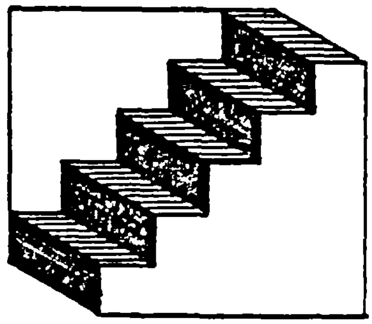

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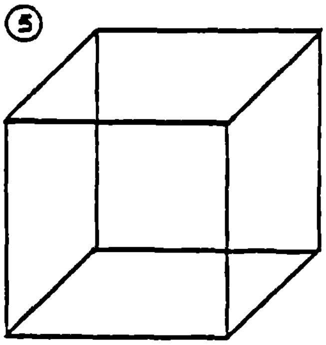


LOOKING AT MORE THINGS

Investigation, discussion, writing. (Visual Perception)

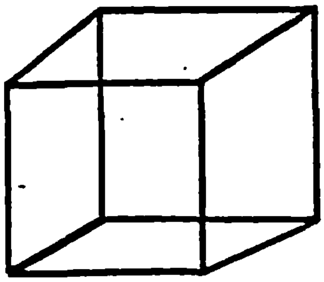
WHAT TO DO In the first part there are two diagrams. If they are cubes, which face is in front? If the other face is in front, what shape is the object?

In the second diagram, is it a drawing of stairs seen from above or are you seeing the underneath of the stairs? What is different about this drawing from the other drawing of some stairs you had before? Turn the drawing on its side - what do you notice?

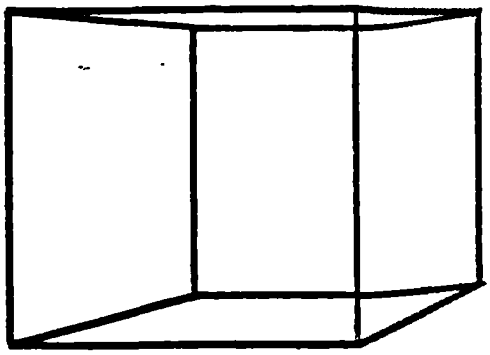
In diagrams 3, 4, 5 and 6, say what is wrong with the diagrams if anything. If nothing is wrong, say so. But look carefully before you answer.

E3 ctd

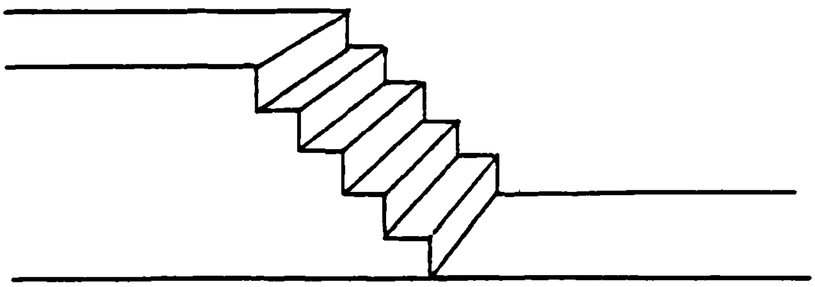
1(a)



1(b)

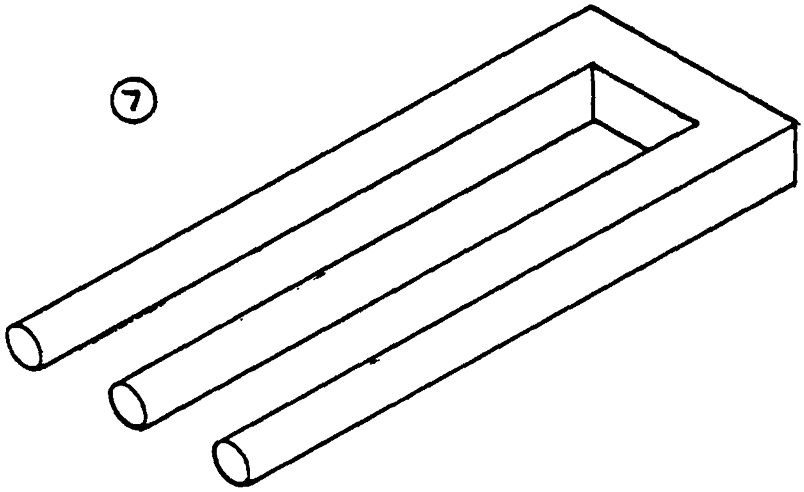


2.

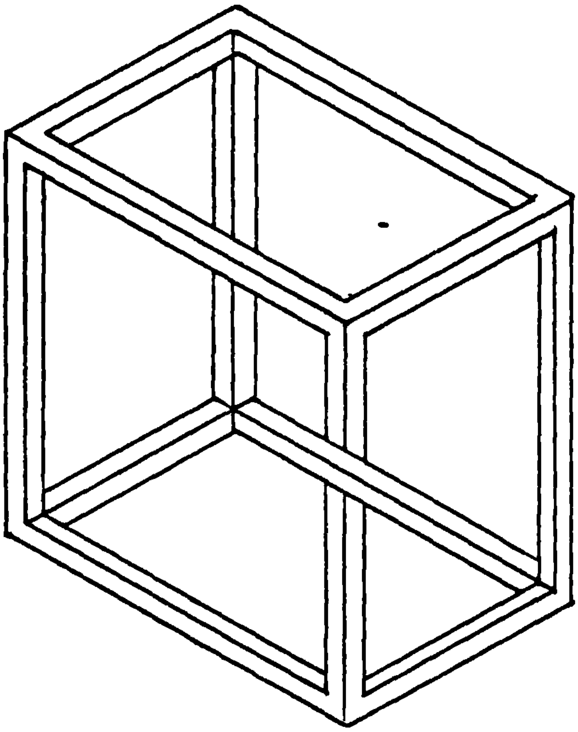


E3 ctd

(7)



(8)



E4

LINES CONVERGING AT A POINT ON THE HORIZON

Structured drawing, (Visual Perception)

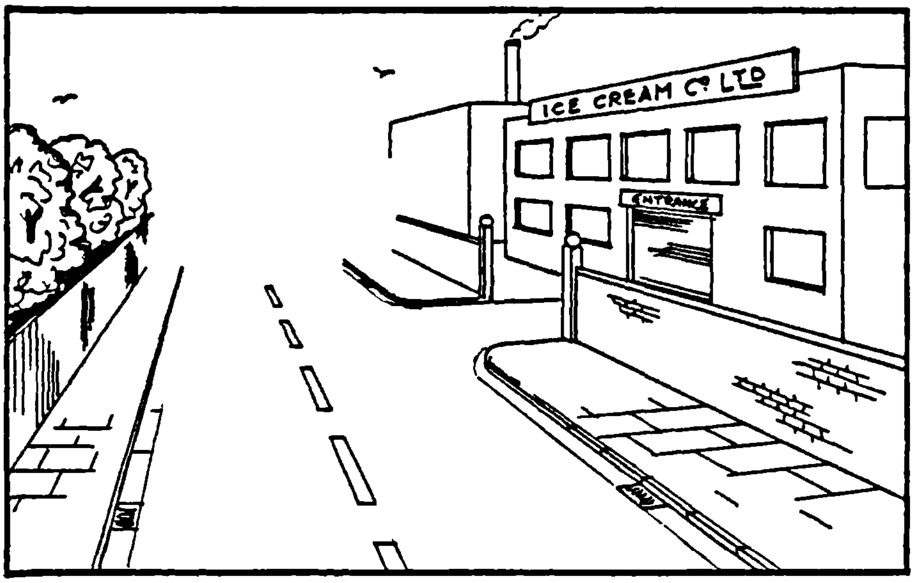
AIM Children have experience in considering the effect of perspective on parallel lines.

WHAT TO DO

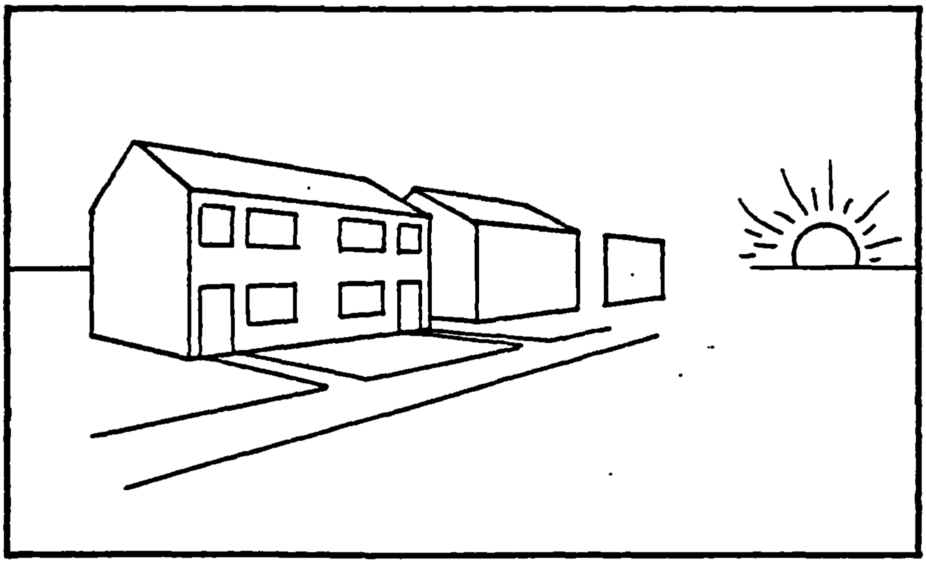
"You (may) have met something like this before. Put in missing lamp posts along the road so they are about the same distance from each other. Also draw in the missing windows on the building further up the road. The picture is not complete - the far end of the straight road has not been put in. Put in the rest of the road and the pavements. Remember the road is straight."

DISCUSSION. Children are likely to bend the lines when they extend them. This gives the appearance of the road going uphill. Discuss the reasons for bending the lines - would they be better just extended? Explain how to draw a perspective view of a chessboard using receding lines in two different directions.

E4 ctd



E4 ctd



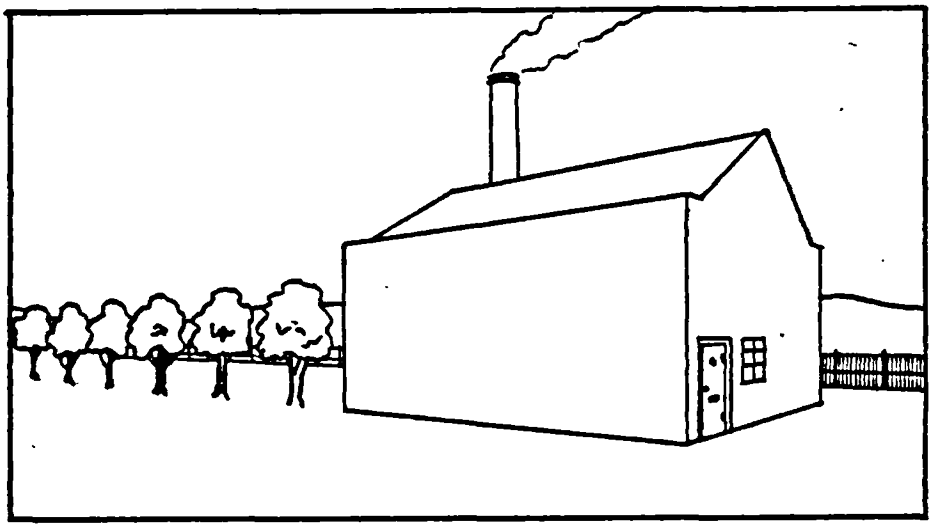
E5

INSERTING MISSING OBJECTS

Structured Drawing. (Visual Perception)

AIM	Children learn to use the ideas of receding lines in a more structured situation.
WHAT TO DO	This is a drawing of a factory and a row of trees. The windows are missing. Draw in the missing windows - about 3, 4, 5 or 6 windows. Make your drawing look right. There is a road missing in front of the factory going on past the trees as well. Draw in the road - you can put in the white lines down the middle. There are 4 lampposts by the side of the road, all the same side of the road, draw in these as well.

E5 ctd



E6

WHERE IS THE MIDDLE?

Selection to be used diagnostically and followed by discussion (Visual Perception)

AIM To discover how children determine distances on perspective drawings. Children learn to draw in the diagonals of the rectangle to find the middle point.

PROCESS Selection, construction.

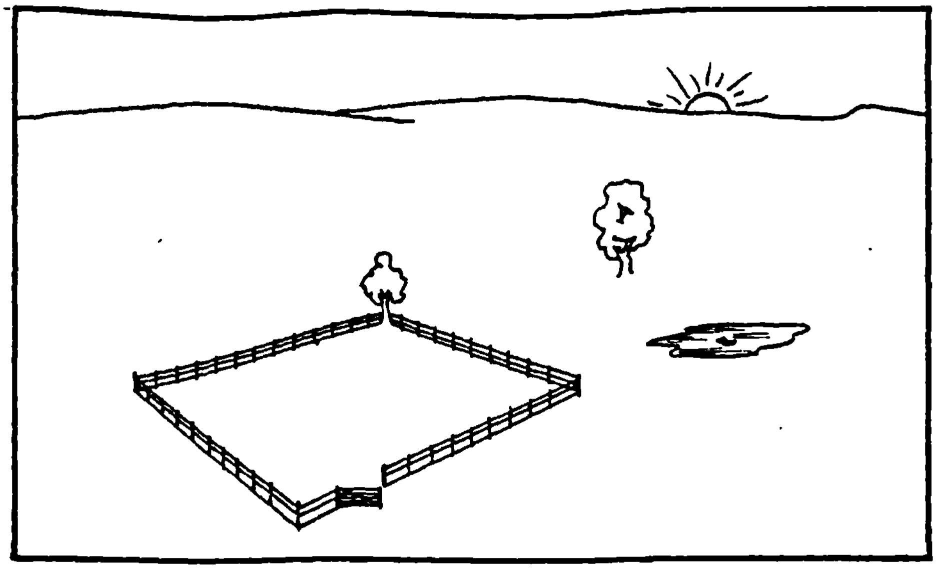
WHAT TO DO

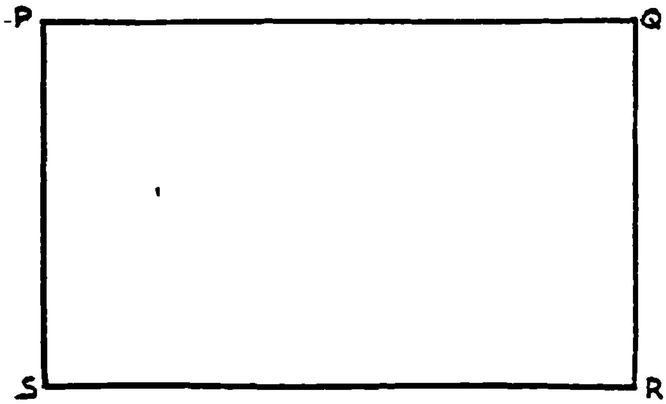
"Here is a square field with the horizon in the distance. It is being seen from a helicopter in the air. Put an X on the corner nearest the helicopter. Put a Y at the centre of the square field.

Here is a drawing of a football field between two fences. You can see the two straight fences disappearing into the distance. Five crosses are marked on the field. Which cross is at the middle of the field? Write down the one you think it is.

Draw in the goal posts on your drawing. Are the goal posts the same size? Answer yes or no. Write down why you think they are, or are not, the same size.

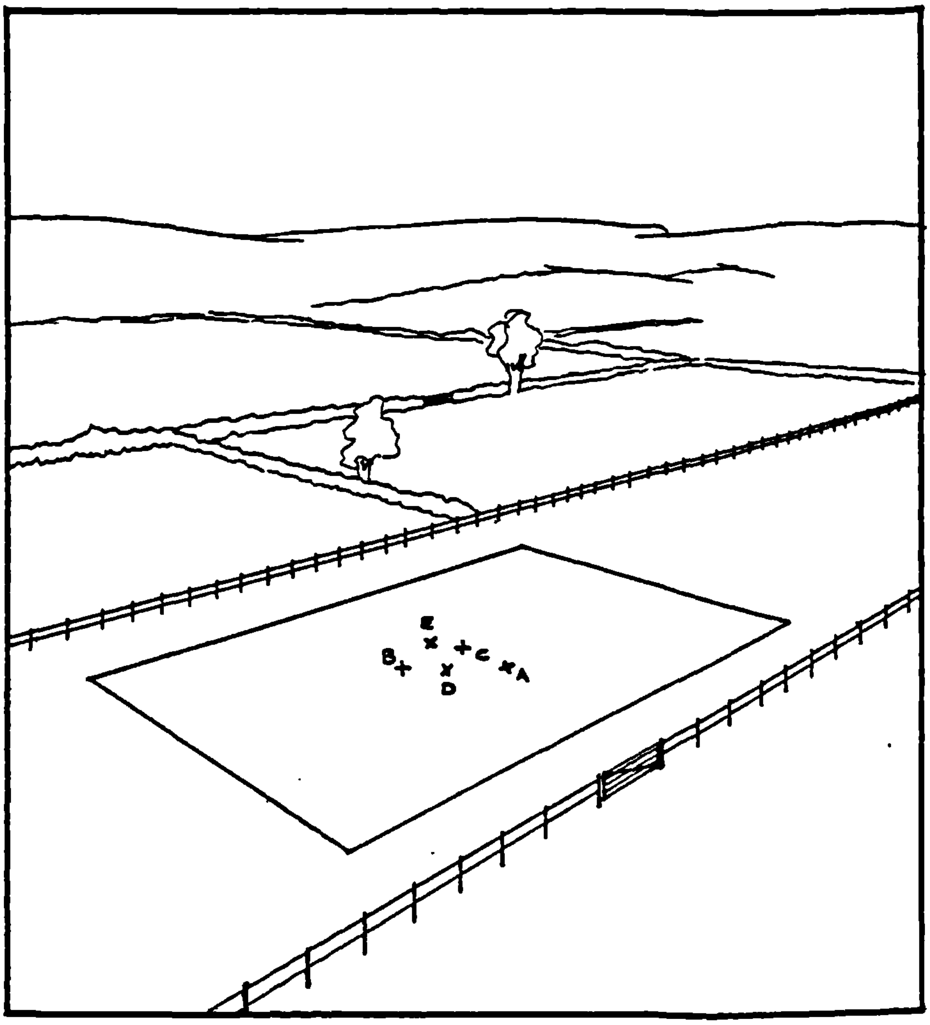
E6 ctd





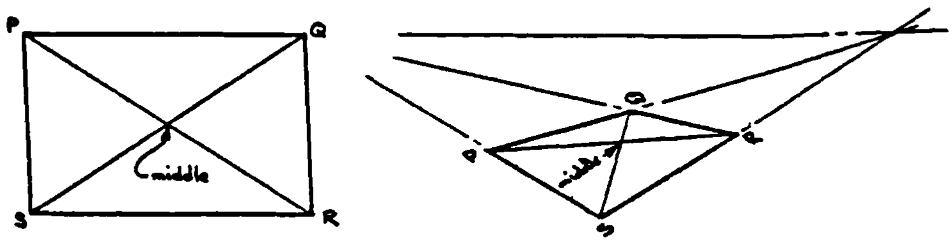
PQRS is a rectangle. How would you find the middle of the rectangle?

E6 ctd



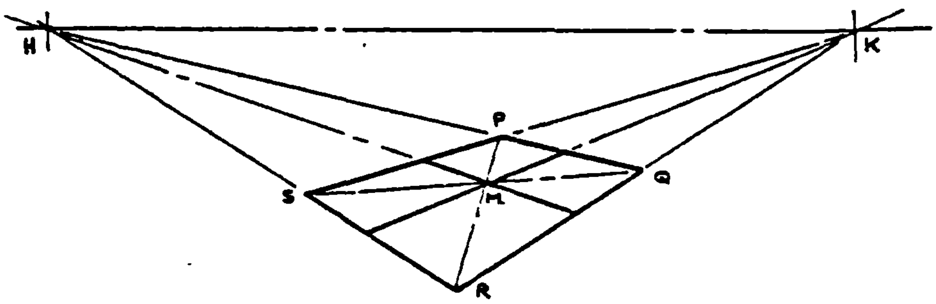
E6 ctd

LEARNING To realise that the middle of a rectangle is obtained by drawing the two diagonals and finding where they meet.



DISCUSSION Ask children for their views about the position of the middle. Give them instructions for constructing the diagonals of a rectangle or square.

ENRICHMENT Some children may use the method to quarter the rectangle as follows:



Construct the diagonals PR and QS to obtain M. Draw HM and KM (H and K being where receding lines meet on the horizon. HM and KM extended quarter the rectangle.

F1

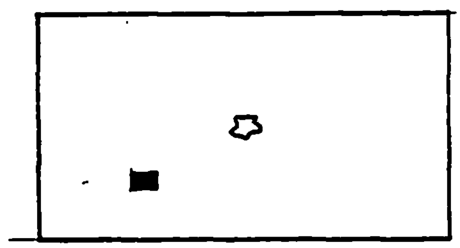
WHERE ARE THEY?

Activity, discussion (Coordination of views)

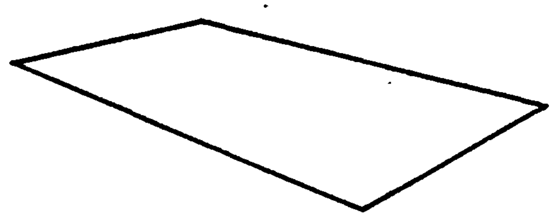
AIM To give children further experiences of imagining a view from a position other than their own in a perspective diagram.

WHAT TO DO

Here is a field seen from above by someone in a balloon.



There is a pond right in the middle. Point to the pond.
There is a swing here too. Point to the swing.
The balloon now floats away from the field and the field looks like this: but the pond and swing are missing.



Put a cross where you think the pond is. Now put a ring where the swing is.

DISCUSSION Children may be asked to draw in the diagonals to find the middle where the pond is, and possibly use the quartering process to find the position of the swing.

F2

MAKING A DRAWING LOOK RIGHT

Structured drawing (Coordination of Views)

AIM To give further experience of receding lines and the representation of near and far objects.

PROCESS Painting, drawing.

WHAT TO DO

Imagine you are the pilot of a Concorde just coming in to land on a runway. There are some airport buildings down one side (like the factories you have seen in other diagrams). Remember that lines recede towards the horizon like they did in the drawing of the street. Draw a picture of the runway and the airport buildings as they would look like from the aeroplane. There are lights by the side of the runway to guide the pilot down in the right direction.

DISCUSSION about the drawings
or
DISPLAY of some of the drawings may be appropriate.

F3

TILED FLOORS AND CHESSBOARDS

Drawing. (Coordination of Views.)

AIM To give some experience of receding lines and the representation of near and far objects.

PROCESS Painting, Drawing.

MATERIALS Pencil, Brush, Paint.

WHAT TO DO Imagine you are lying on the floor. The floor is black and white tiles like a chessboard. Draw a picture of what the floor would look like from down there. Afterwards get down there to see how good your drawing was. Look at a chessboard from an edge. Draw a picture of what it looks like from very close to an edge.

MOTIVATION Present as a challenge.

LEARNING That objects further away are drawn smaller. That receding parallel lines appear to converge.

DISPLAY Of some of the drawings.

VOCABULARY Near, far, closer, big, small, bigger, etc.

ENRICHMENT Various exercises of a similar sort.

EVALUATION Children are questioned - does it look right? Which tiles are farther away? Do the squares look like squares?

MORE CHESSBOARDS

Structured Drawing (Coordination of Views)

AIM To focus children's attention on constructing receding lines.

PROCESS Drawing.

WHAT TO DO

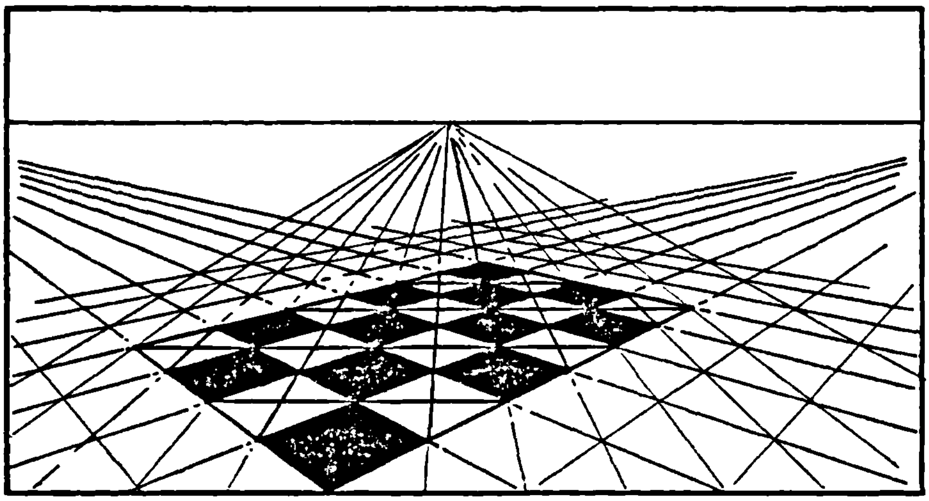
Imagine you are a toy soldier or a doll standing on a corner of a chessboard. The straight lines on the chessboard will be receding into the distance and the further away the lines are the closer they appear to be. The squares on the far side of the board will appear smaller than the near ones.

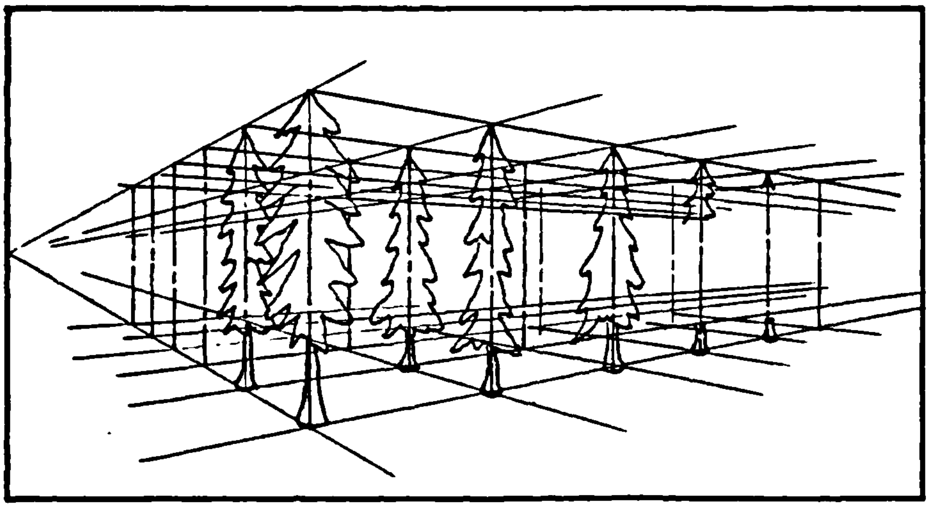
Draw a picture of the toy's or doll's view of the chessboard. Remember to make receding lines converge on the horizon.

ENRICHMENT
DISCUSSION

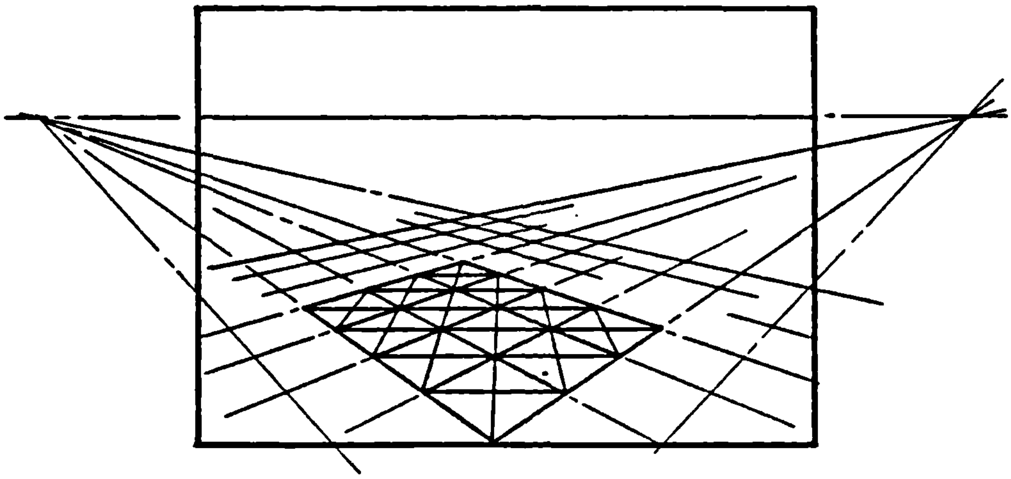
Look at a chessboard close to; near a corner. Does your drawing "look right"? How could it be made to look better?

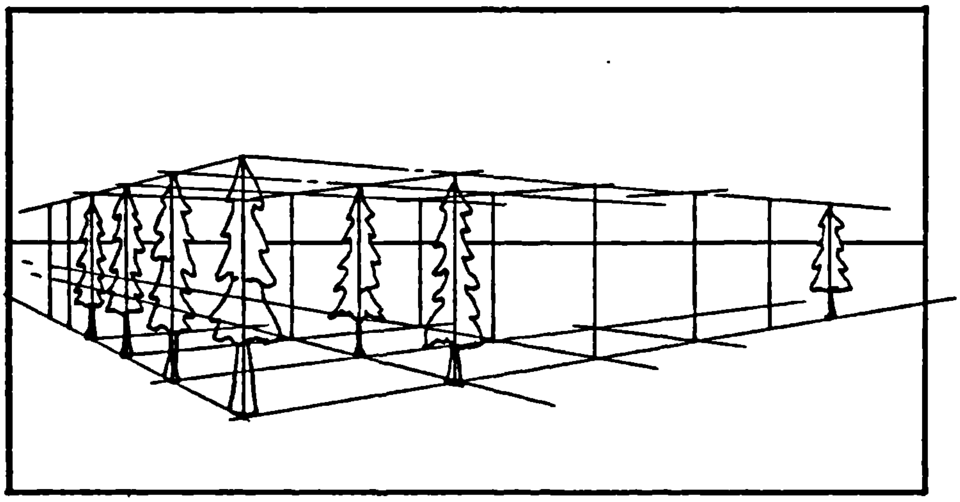
F4 ctd





F4 ctd





F5

MORE DRAWINGS AND PAINTINGS

Structured drawing. (Coordination of Views)

AIM — To give children experiences in considering an object from different positions or viewpoints.

PROCESS — Painting, drawing.

MATERIALS — Pencil, paint, brush, crayon.

WHAT TO DO — Draw a picture of: a table, a house, a field or a classroom any way you like. Now draw what it looks like from: in front, from above, or from another place.

MOTIVATION — What do you think a table (or others) looks like? What if we want to make it look real? Will it look different from in front? Why? How will it be different? Imagine taking a photograph. What will it look like from up here? Come and see. What do you notice?

DISPLAY — If appropriate.

ENRICHMENT — Choose your own object now and draw or paint this.

MINIMUM EXPECTED OF CHILDREN — That the child will attempt to make the drawings different, that he can explain why they should be different in very general terms.

EVALUATION — The child should have 'improved' his drawing from those where less guidance and discussion was given.

COMMENT — This is a very difficult exercise. It may well be left until very much later. The problem of coordinating a variety of perspectives would defeat many adults. It may feedback useful information about their thought processes.

F6

WHERE ARE YOU?

Activity, selection, discussion. (Coordination of Views)

AIM To give children experiences of determining a different viewpoint. To discover whether they have the capacity to do so.

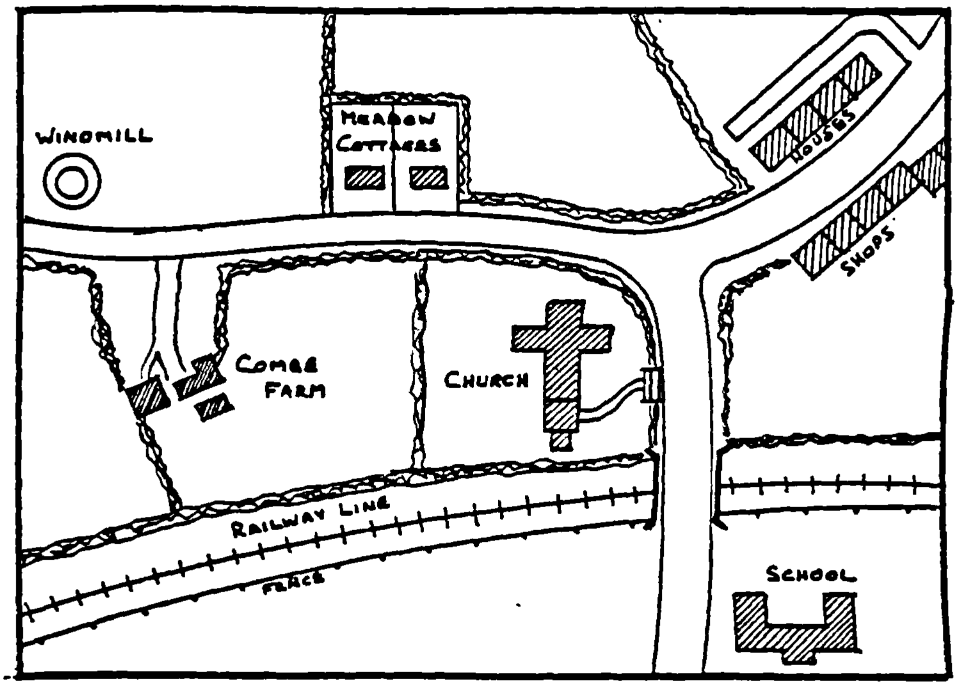
PROCESS Selection.

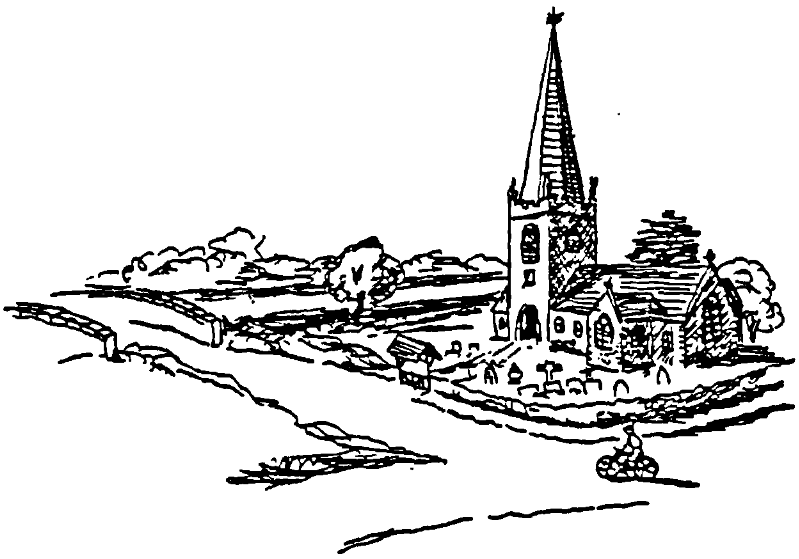
WHAT TO DO

"Here are two diagrams, the first one is a plan, a map of a village. Point to the windmill, to the school, the church, the shops. Where is the bridge? Why is there a bridge there?

The other diagram is a drawing of part of the same village. You are looking at the church from somewhere in the village. Point to the bridge over the railway, the lychgate and the path leading to the church. Now where is the bridge on the map? Put a cross on the map from where you think the picture was drawn. Where, on the map, would you be standing to see the drawing like it is, with the church there, and the bridge and the gate and the path as they are?"

F6 ctd


WINDMILL
MEADOW COTTAGES
HOUSES
SHOPS
COMBE FARM
CHURCH
RAILWAY LINE
FENCE
SCHOOL



F7

WHICH DRAWING IS FROM WHICH PLACE?

Multichoice selection, (Coordination of Views)

AIM Children learn to imagine the view from a position different from their own.

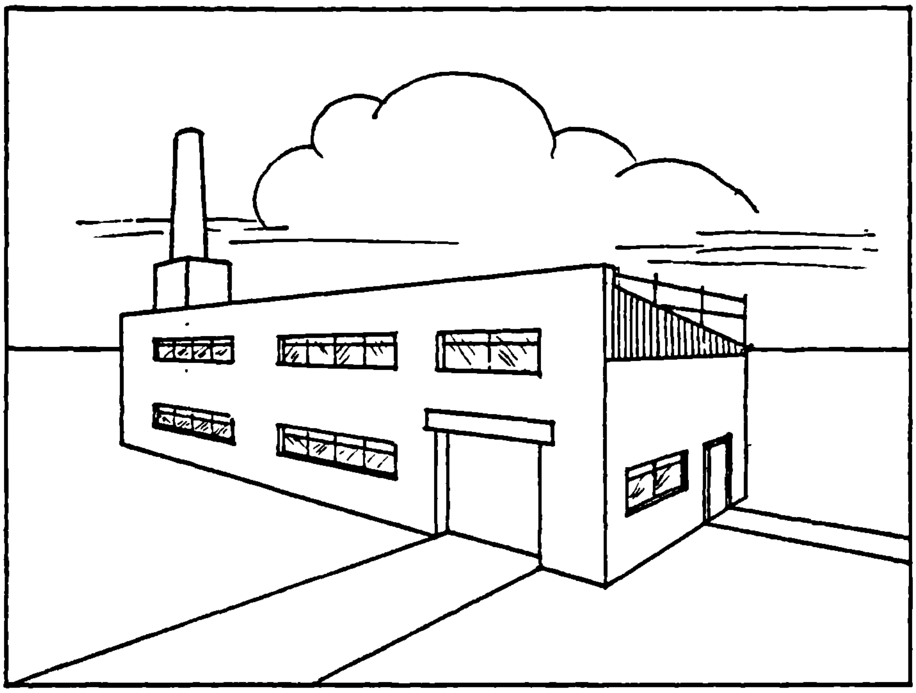
PROCESS Selection.

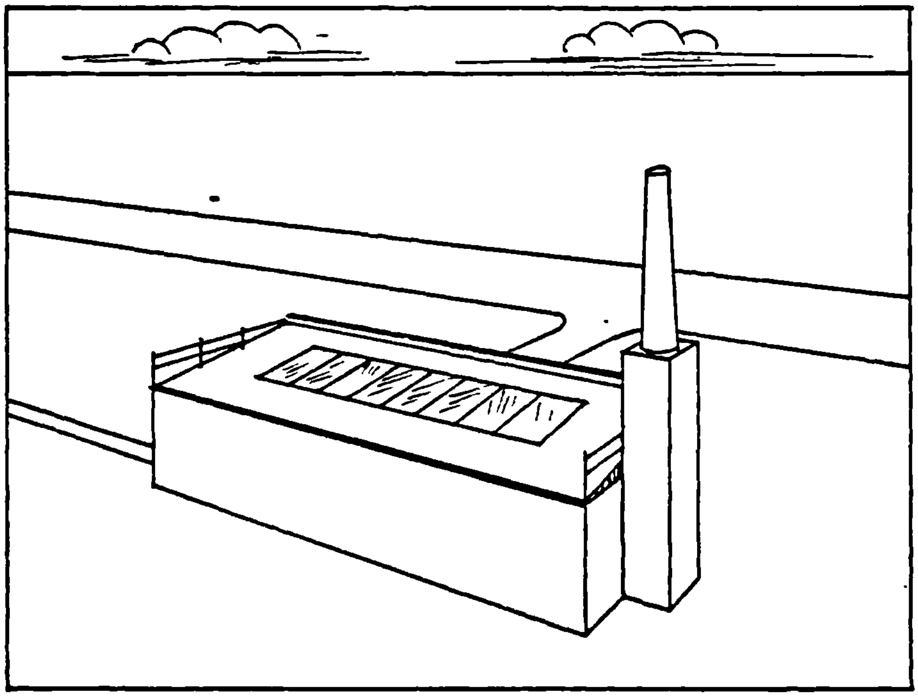
WHAT TO DO

Here is a drawing of a factory. The back wall is blank without any windows. It is very like one you have seen before. There are also five diagrams marked 1, 2, 3, 4 and 5.

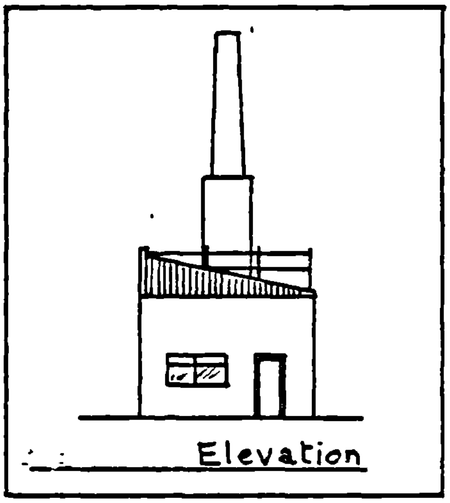
Which is the diagram which is the plan of the factory - the view from A? Which is the front elevation - the view from C? Which is the side elevation - the view from B? Which is the back view of the factory? Which view is the odd one out - it is not a view of the factory at all?

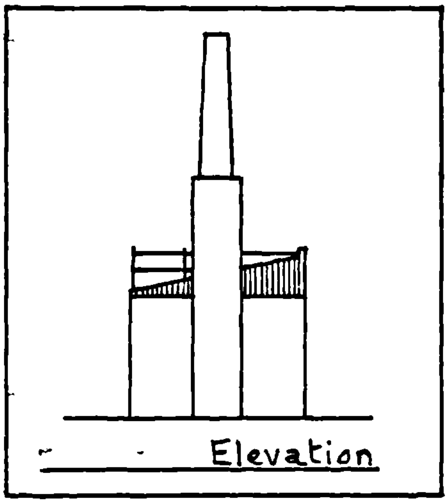
F7 ctd

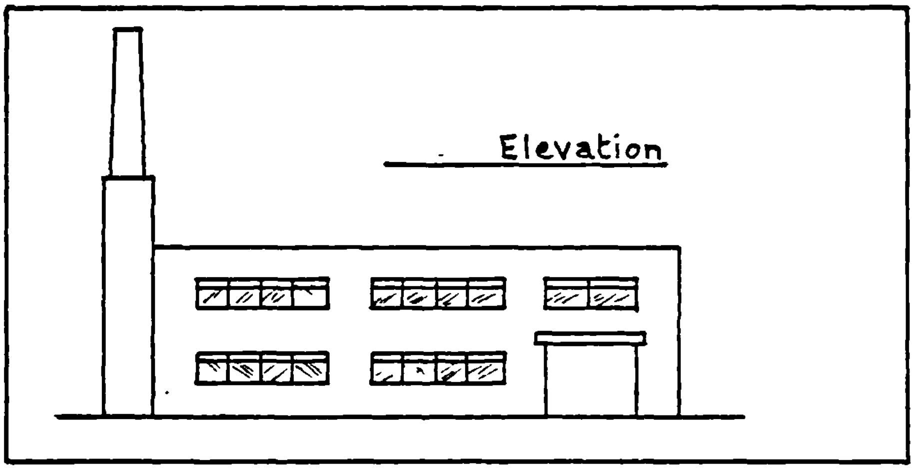


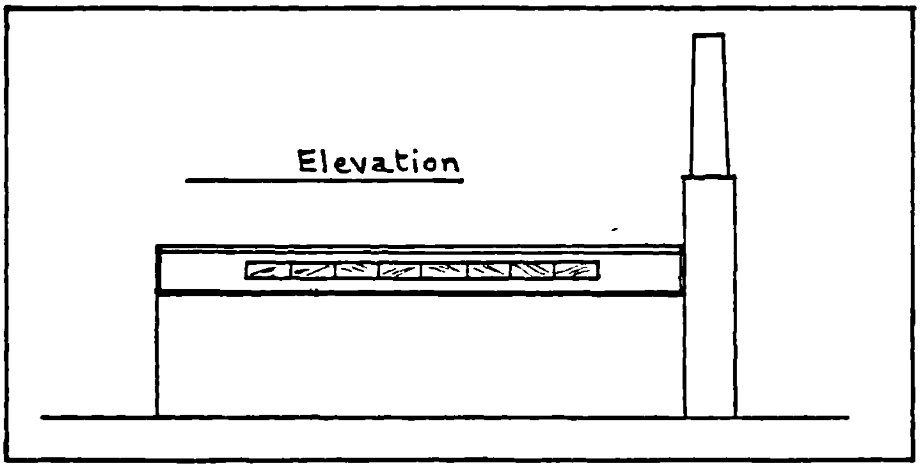


F7 ctd

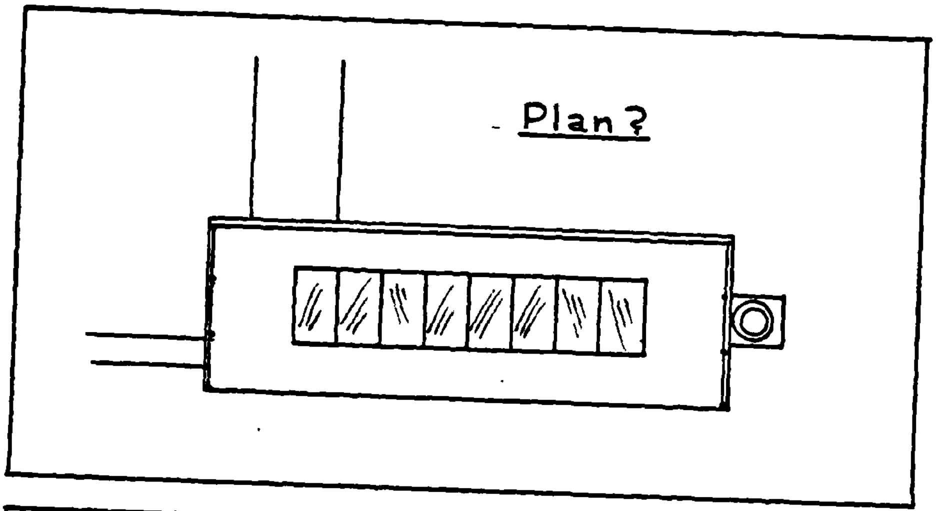


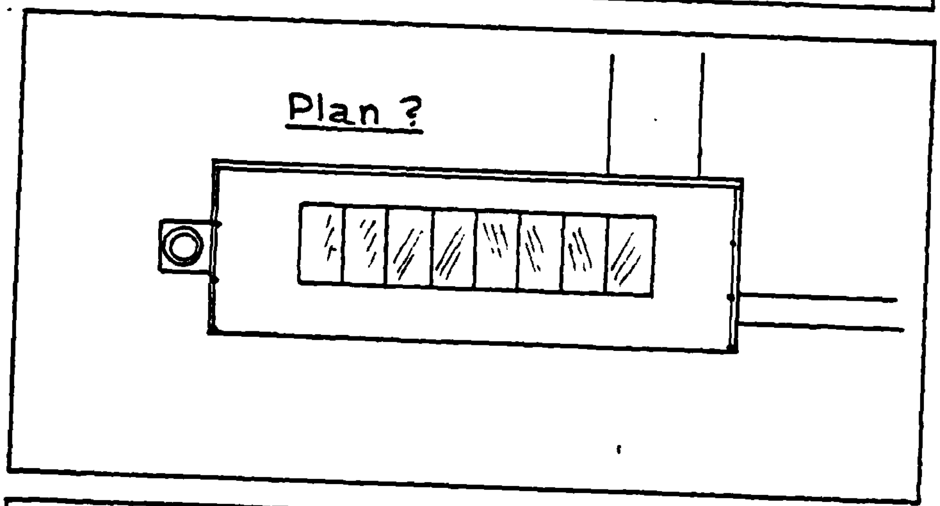


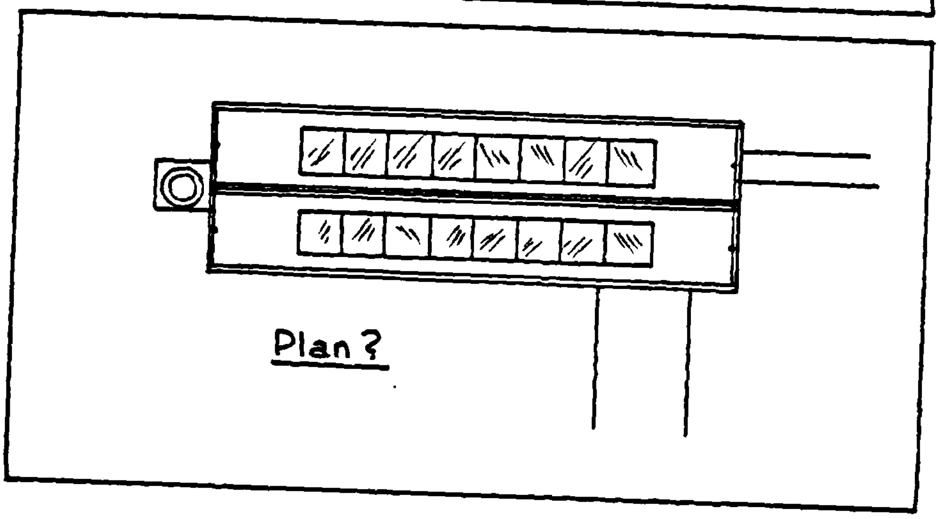




F7 ctd







MAKING MORE DRAWINGS LOOK RIGHT

Structured drawing, (Coordination of Views)

AIM To provide children with experiences of drawing a perspective view given a two dimensional affine (isometric) representation.

PROCESS Structured drawing.

MATERIALS Isometric paper. Projective 2-D dotty paper. Projective 2-D lined paper. Projective 2-D dotty lined paper.

WHAT TO DO

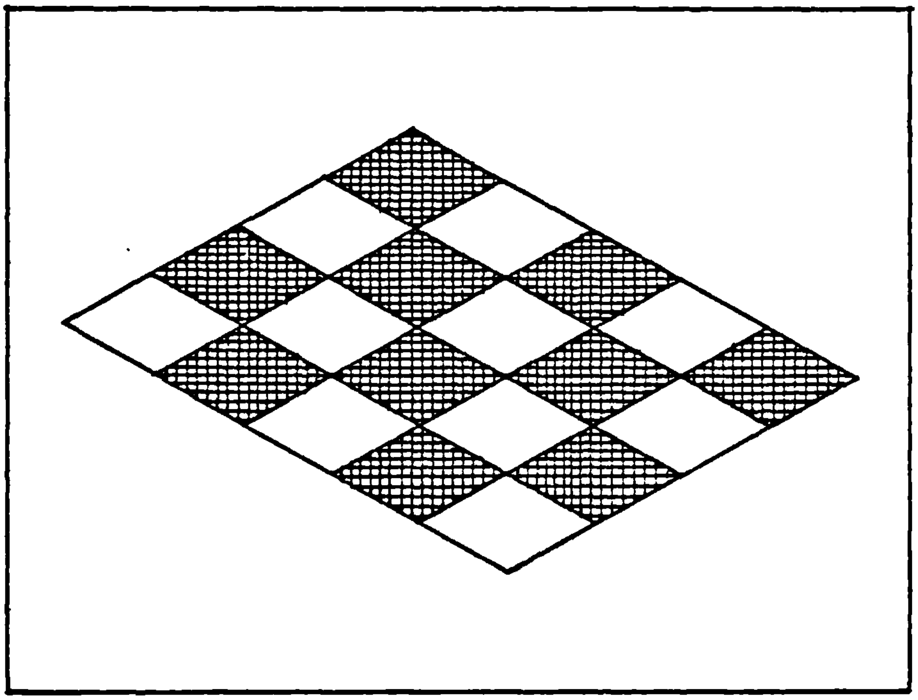
"Here is a drawing of some square fields. You will see that they have not been drawn properly because the receding lines do not converge towards a point on the horizon. Try to draw the fields so that the drawing looks right. If you find this too difficult, some paper with lines on or dots on may help.

When you have drawn your fields shade in the squares in two different colours like a chessboard."

ENRICHMENT

Some children may be encouraged to draw the diagonals of the squares. If they do it carefully these parallel diagonals also converge to a point on the horizon.

F8 ctd



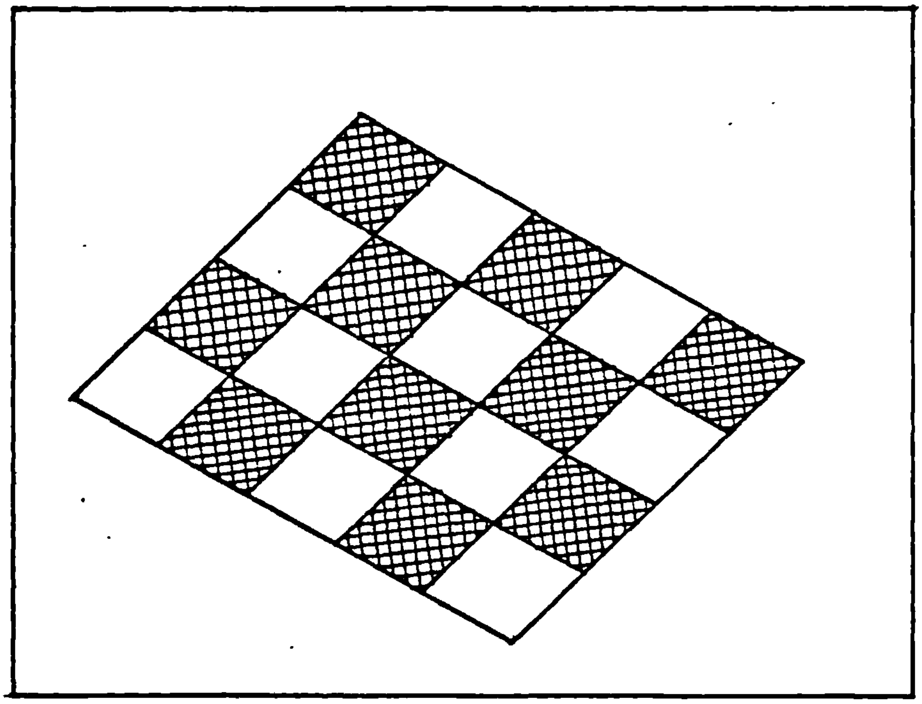
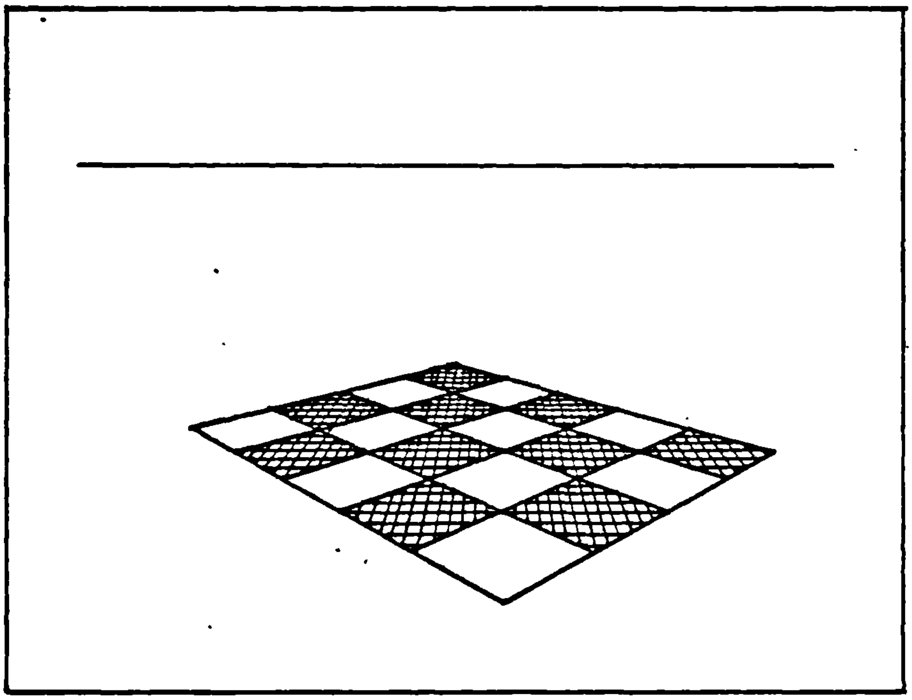
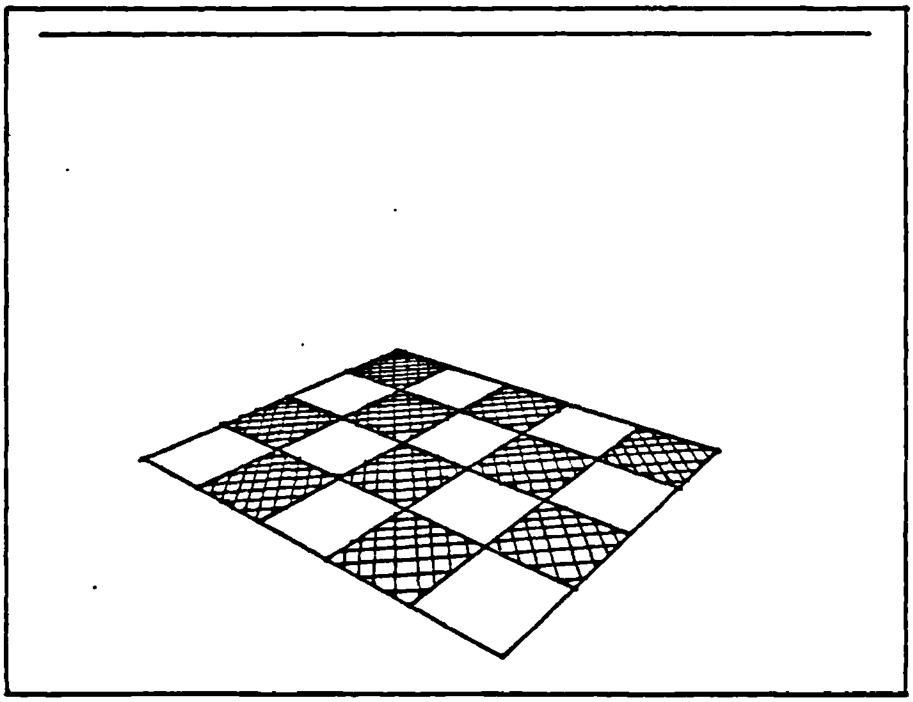
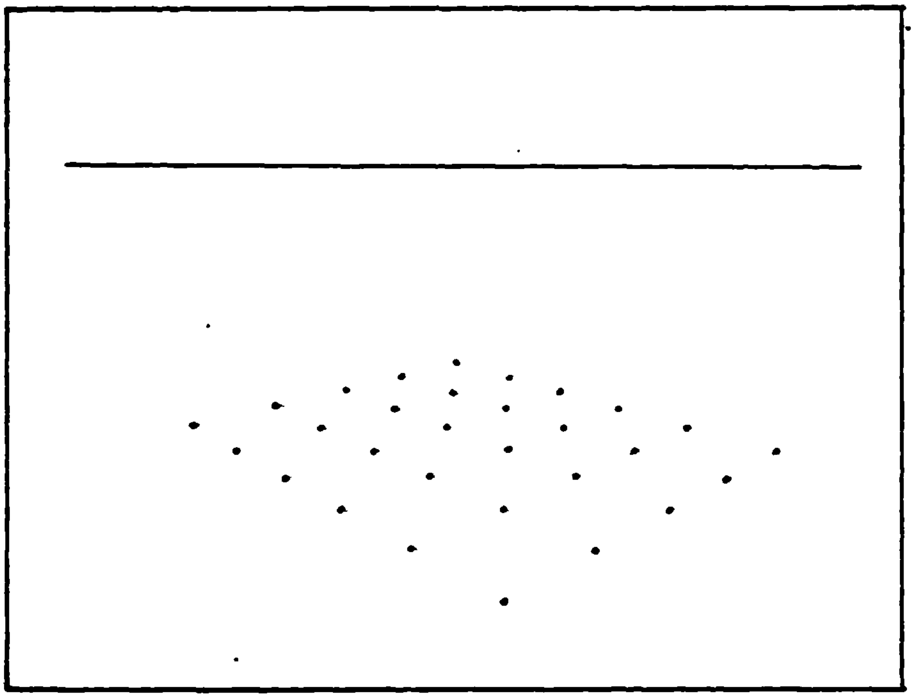


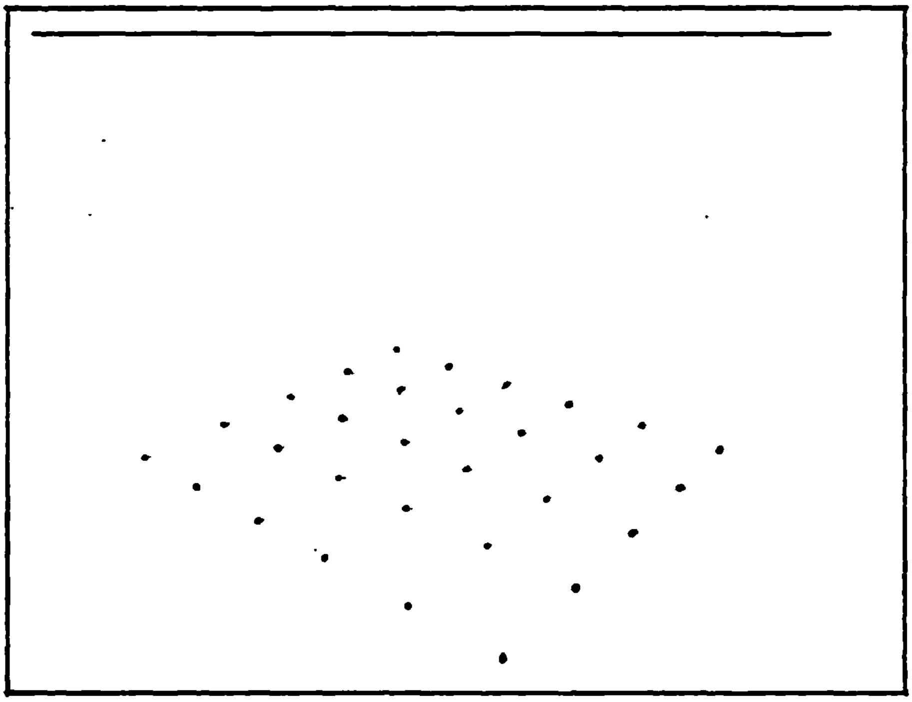
Fig ctd



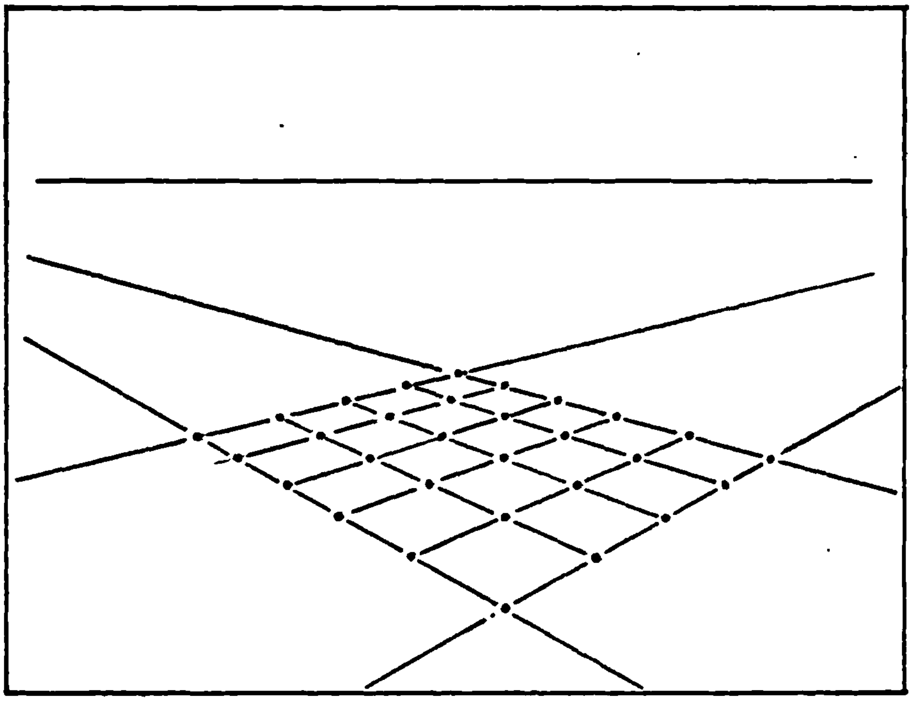


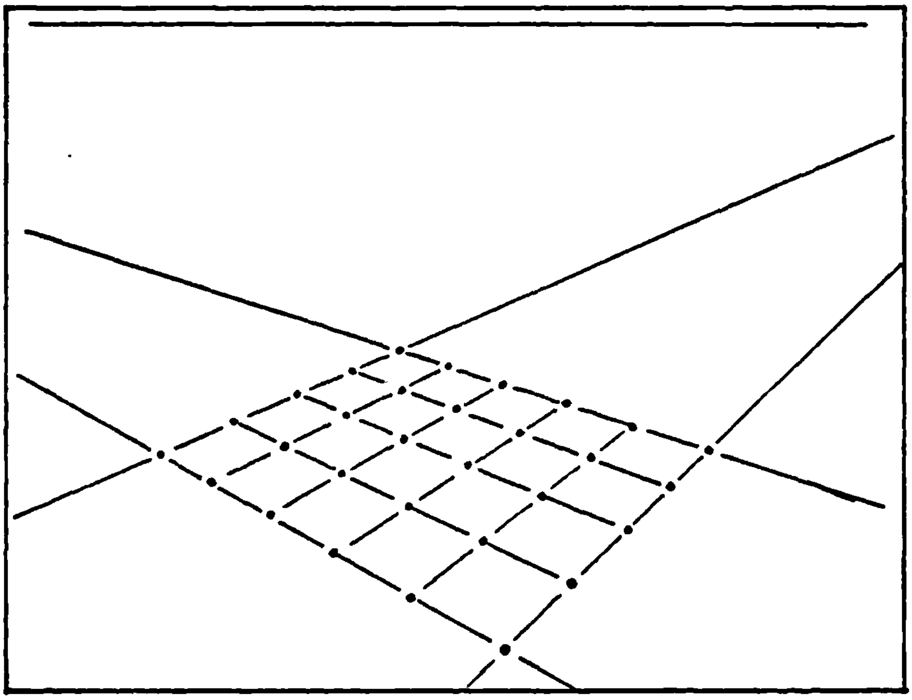
F8 ctd



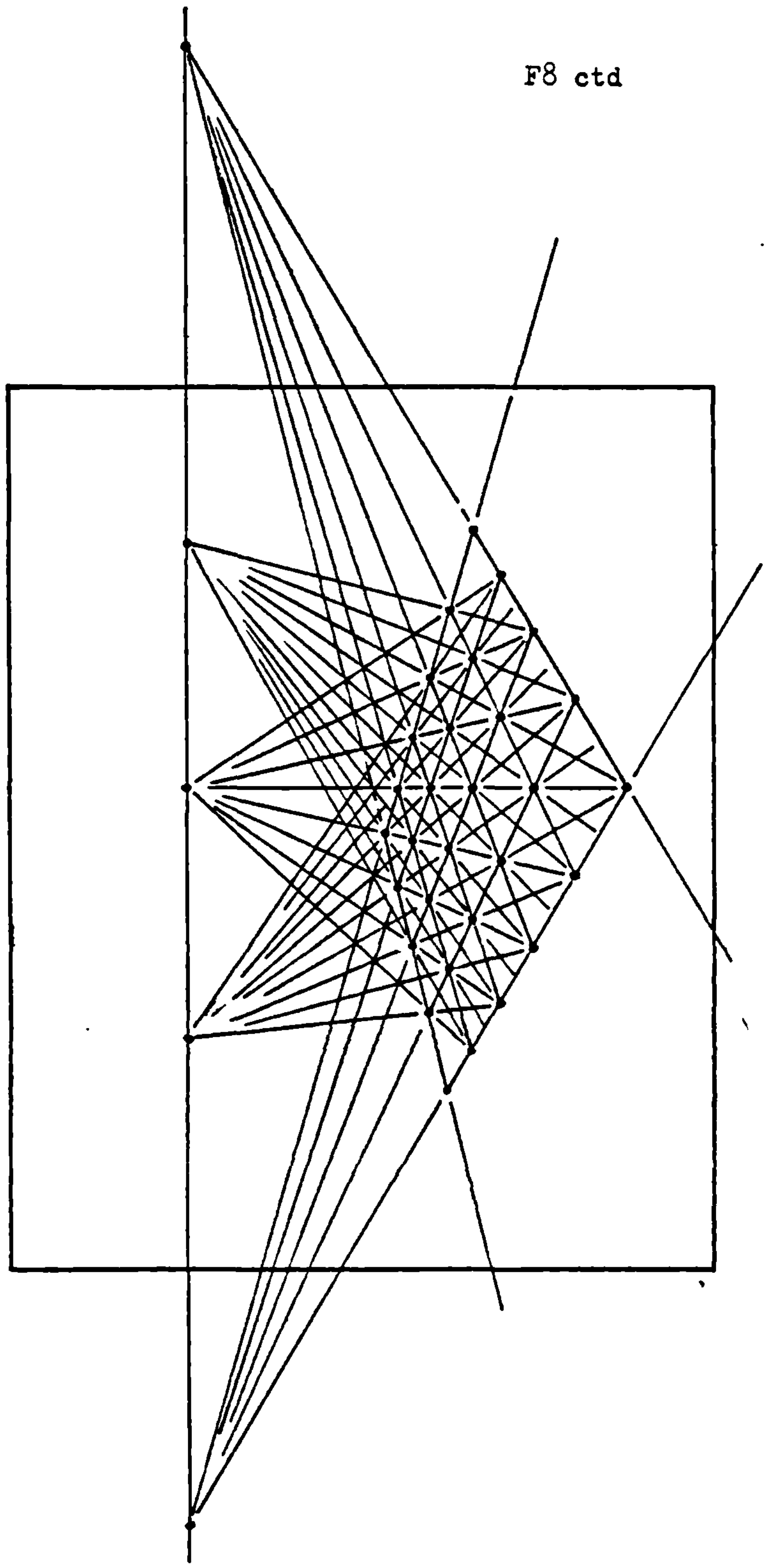


F8 ctd

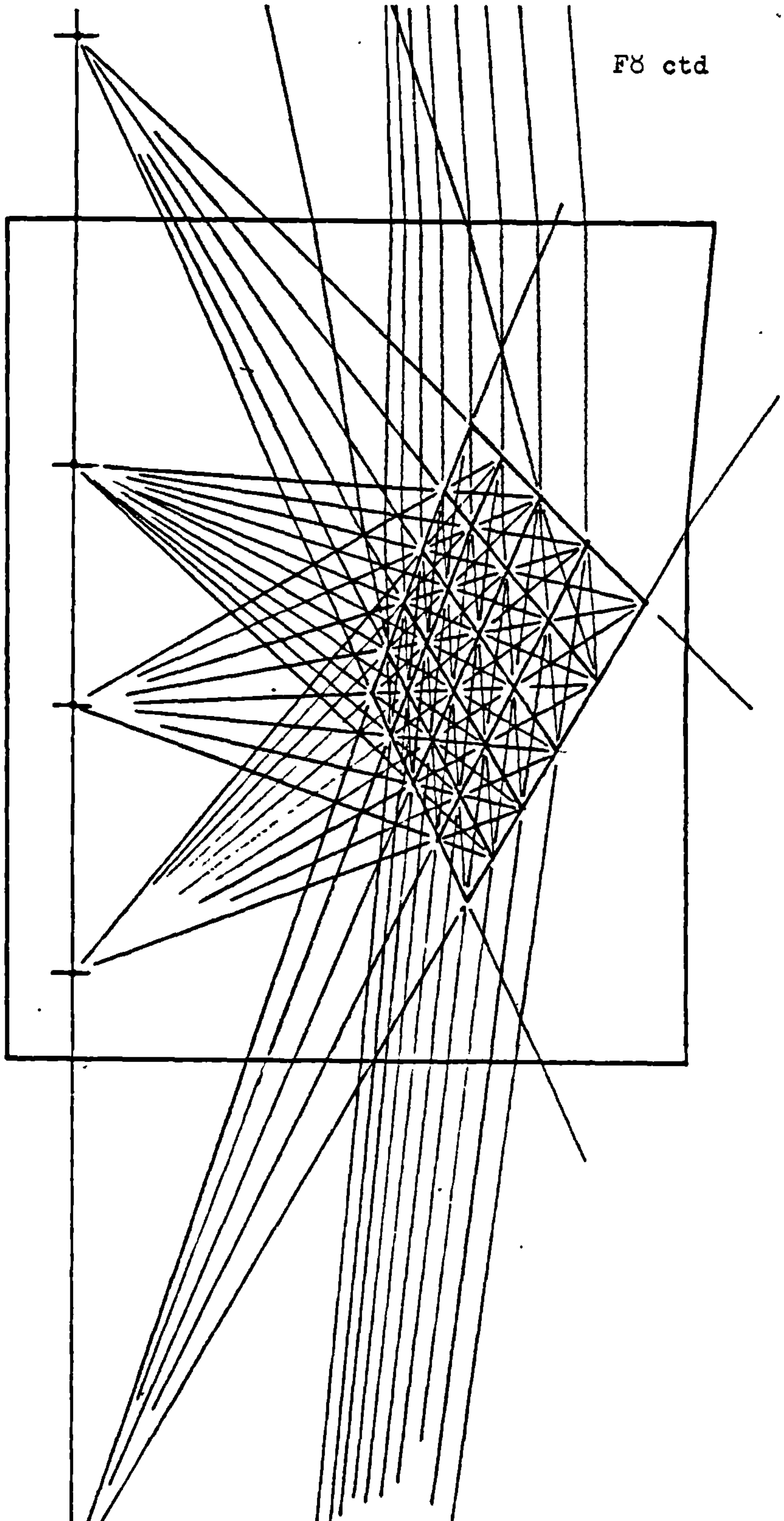




F8 ctd



Fȣ ctd



F9

DRAWINGS OF LARGE CUBES

Structured drawing, (Coordination of Views)

AIM To provide children with experiences of drawing perspective views of three dimensional objects.

PROCESS Structured drawing.

MATERIALS Isometric paper. Projective three dimensional lined, dotty and dotty-lined paper. Projective two dimensional, lined dotty and dotty-lined paper.

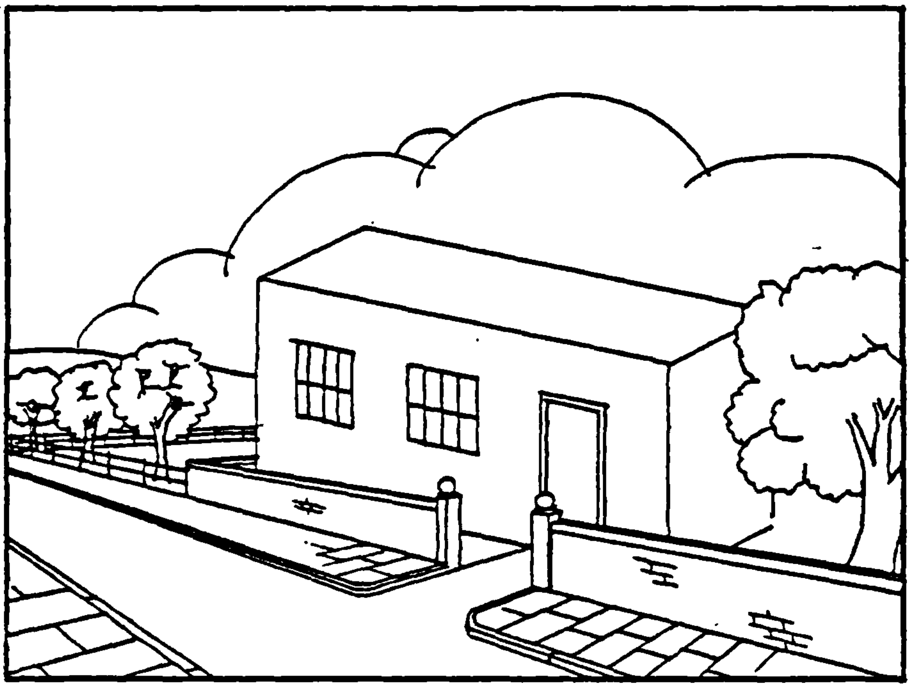
WHAT TO DO

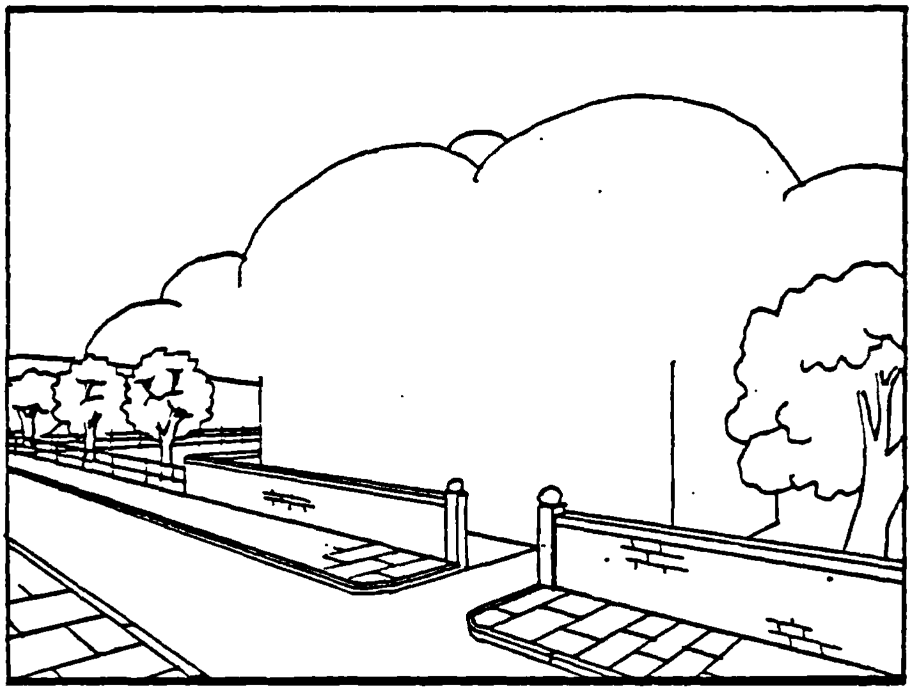
"Here is a drawing of a building made out of large cubes. You will see that it has not been drawn properly because the receding lines do not converge. You can probably see that the far end of the cubes seem to be too tall and the roof of the cubes seem to be sloping, not flat.

Try to draw the cubes building so that it looks right - so that the receding lines converge."

"If you find this too difficult, use some of the paper with lines or dots on it."

DISCUSSION
DISPLAY of some of the attempts.





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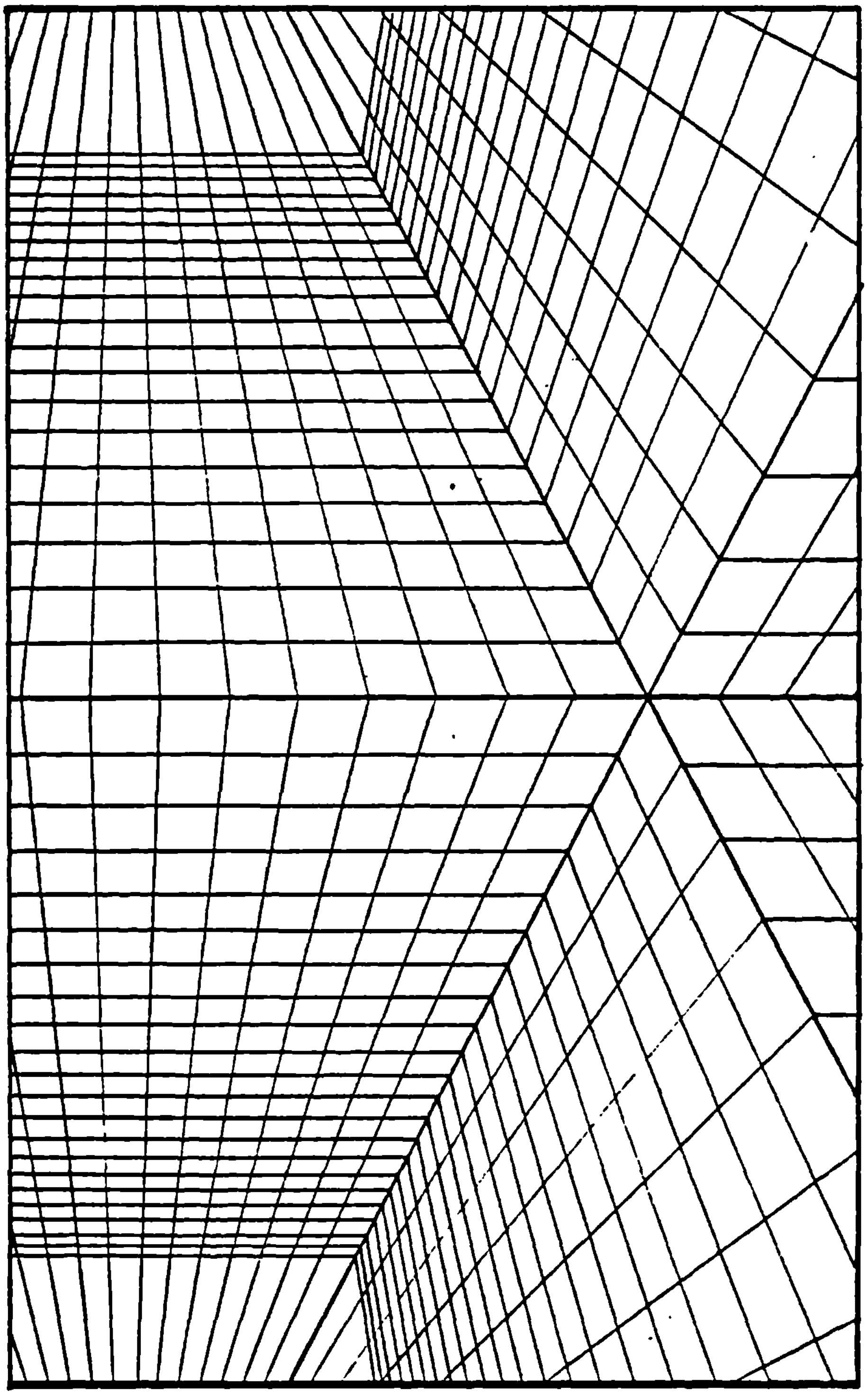


F10

WHAT WILL IT LOOK LIKE FROM OTHER PLACES?

Multichoice selection, (discussion) (Coordination of Views)

AIM To give children further experiences of different viewpoints.

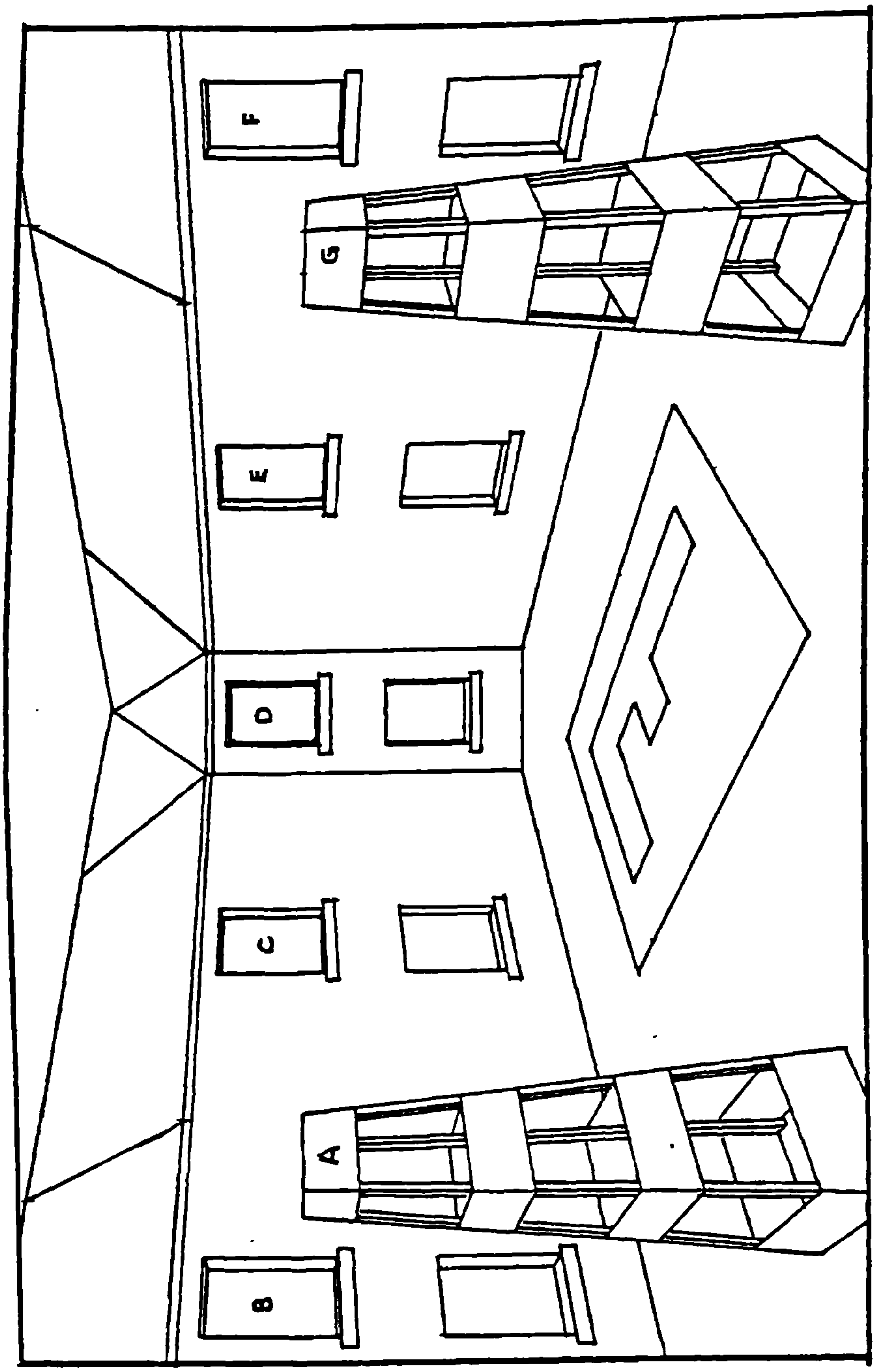
PROCESS Selection, discussion.

WHAT TO DO

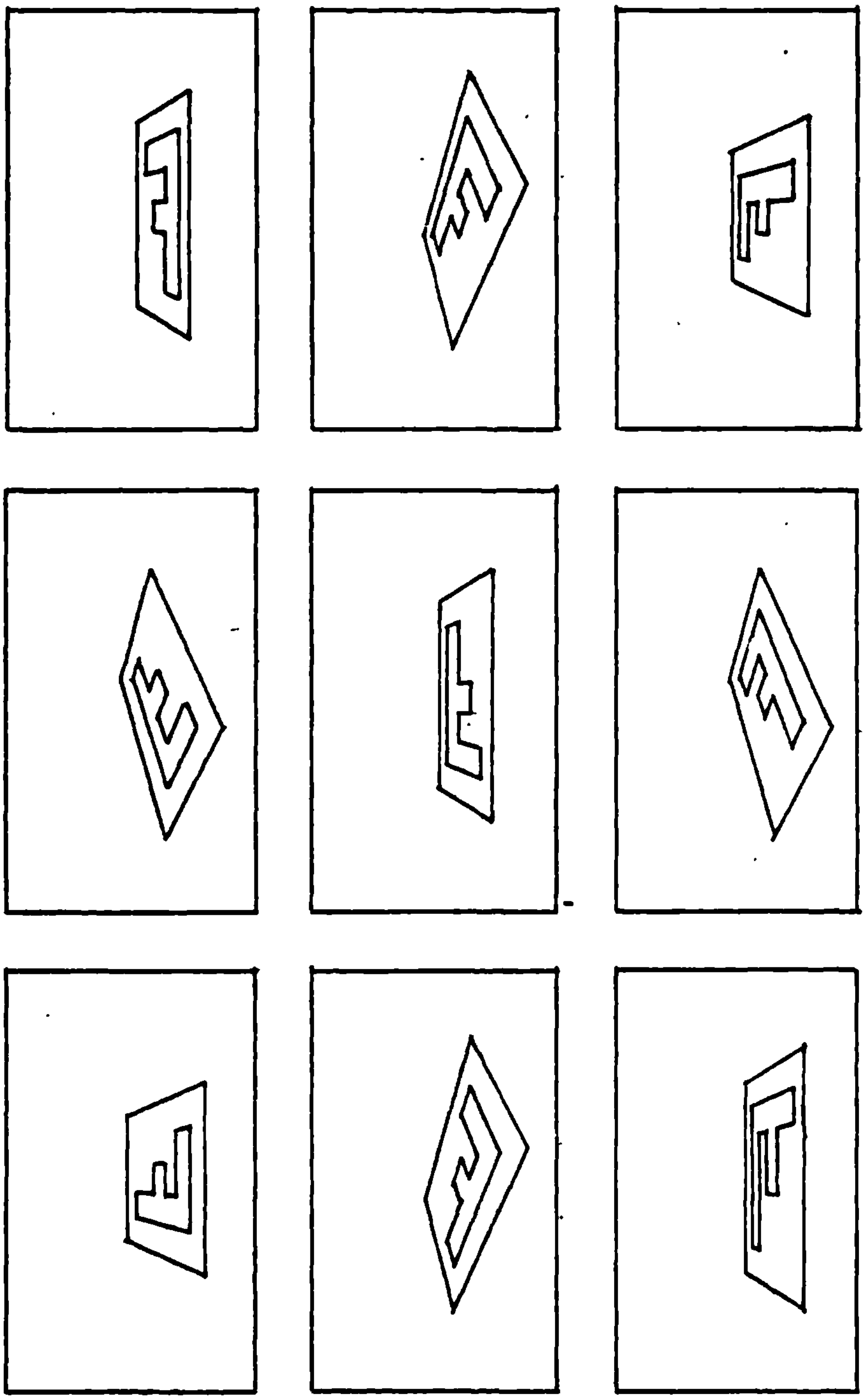
"Here is a drawing of a school playground. On it is marked a large diagram like a backward capital F.

From different parts of the playground the diagram will look different. For each of the diagrams decide from which position you are looking."

F10 ctd



F10 ctd



APPENDIX TWO

Getting Geometry into Perspective : A Bird's eye View

A Geometrical Hierarchy of Transformations of a Figure in a Plane

Names of groups of transformations

(a) Identity

(b) Translation

(c) Direct congruence

(d) Congruence (direct and indirect)

(e) Similarity

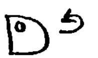
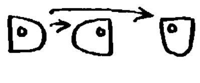
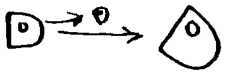
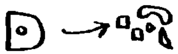
(f) Affine

(g) Projective

(h) Topological

(i) ?

GEOMETRICAL HIERARCHY OF TRANSFORMATIONS OF A FIGURE IN A PLANE

DESCRIPTION OF THE TRANSFORMATIONS	DIAGRAM SHOWING TYPICAL TRANSFORMATION(S)	NAME OF TRANSFORMATIONS	NAME OF GROUPS OF TRANSFORMATIONS		PROPERTIES RESERVED IN THE TRANSFORMATION
The object remains where it is		Identity	Identity	Truly equivalent	Position & those below
The object is moved parallel to itself		Translation inc. identity	Translation	Position no longer preserved	Orientation & those below
The object is rotated in the plane		Rotation inc. translation	Direct conguence	Orientation lost as well	Sense & those below
The object is relected or glided i.e. turned over		Indirect congruence (+ direct)	Congruence (direct & indirect)	Sense lost	Size & those below
The object is dilated - enlarged or shrunk		Similarity inc. congruence	Similarity	Size lost	Shape & those below
The object is sheared sideways		Affine inc. similarity	Affine	Shape lost inc. angles	Parallism & those below
The object is projected like map projections		Projective inc. affine	Projective	Parallelism lost	Straightness & those below
The object is distorted topologically (a 1-1 bi-continuous change)		Topological inc. projective	Topological	Straightness lost	Continuity & those below
The object "explodes"		?	Anarchy	Continuity lost	Everything?

(Although a figure has been transformed in the above diagrams, it is in fact the whole plane which is being transformed e.g. in topological movements the plane is deformed in such a way that a particular figure has the same essential topological properties.)

APPENDIX THREE

Geometrical, Spatial Experiences for 5 to 12 year olds

Discussion Document

(a) Initial ideas

(b) Early spatial ideas

(c) Pictorial spatial ideas

(d) Solid shapes (3D)

(e) Plane shapes (2D)

(f) Tessellations

(g) Further plane shapes

(h) Symmetry

(i) Similarity (Keeping the same shape)

(j) Further solid shapes

(k) Vertical, horizontal, parallel, angles, turns, directions

(l) Translations

(m) Co-ordinates

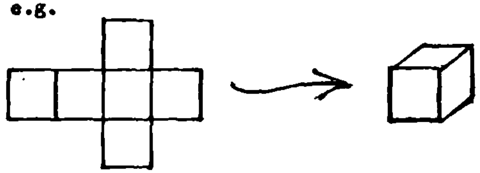
(n) Enrichment

Andrew J SALISBURY

GEOMETRICAL, SPATIAL EXPERIENCES FOR 5 TO 12 YEAR OLDS

Topics Concepts Skills	Activities	Language	Recording
1. Initial ideas	Collecting, discussing, sorting, handling collections of natural and manmade objects - all three dimensional (3D). The naming of these shapes is not vital at this stage.	as appropriate	Use of display with objects and cards naming them.
2. Early spatial ideas	Free play construction with e.g. Lego, Meccano, building bricks and rods, poleidoblocs, plasticine. Children's diagrams of how they come to school, etc. How they picture the playground, classroom, etc. Verbal and picture descriptions of journey, the London Underground map or similar to describe and plan journeys.	in, on, by, near, closed, open curves left, right, inside, outside, between, below, above, next to, joined to.	The activities themselves are a record. Display children's diagrams.
3. Pictorial spatial ideas	If possible use an aerial view of the school and neighbourhood. "Can you recognise this?" "Where do you live?" etc. Drawing views from classroom window. Studying aerial views of New York or London or similar. "What is this building?" "How would go from....to..? Lining up posts or rods, in straight lines by eye. Forming straight lines by stretching string and by paper folding.	straight, curved, round, up, down, view, see.	Drawing picture maps - showing how the child sees spatial relationships. Do not over-criticise but ask questions "Why did you put the shop there?" etc.
4. Solid shapes (3D)	Collecting, sorting, describing verbally, recognising shapes. Model making using plasticine, cardboard. Use materials to construct a shape from a picture of it. (Names to be given but not insisted upon).	cube, cuboid (shoe box shape), sphere disc, cylinder, pyramid, cone, tetrahedron, edge, face, big, small, corner (vertex), thick, thin, flat, surface.	Of the type:- "I made a cylinder, it looks like this..." etc "We sorted our shapes. We put......and...... together" etc.

Topics Concepts Skills	Activities	Language	Recording
5. Plane shapes (2D)	Cardboard shapes, e.g. triangles, squares given to children. Looking for these shapes in solids, in the environment. Shapes of common objects, e.g. stamps, road signs, buildings, fields etc. "Is a triangle a triangle when it's upside down?" (Not all triangles should be equilateral - use other shapes). (Keep the word rectangle to include squares and oblongs).	triangle, square, oblong, equal, circle, side.	drawing shapes and labelling e.g. big, number of sides.
6. Tessellations	Floor tiles, mosiacs, patterns with squares, oblongs, equilateral triangles, any triangles - giving shapes to the children preferably in three or four colours. Deciding which shapes fit together, if they are all the same size and shape as each other. "Do equilateral triangles? any triangles? squares? oblongs? diamonds? quadrilaterals? regular hexagons? (All yes). Do circles, regular pentagons, 50 pence pieces? (All no). Tessellating using potato prints and similar (Building with cubes, cuboids, tessellations in 3D e.g. looking at bricks in walls - different bonds).	tessellate, pattern tile motif design, fit together, without gaps. repeating design along a line repeating design all over a plane. hexagon, pentagon, diamond, (rhombus) quadrilateral.	displaying child's work, patterns, and writing about it. the activity is usually its own recording.
7. Further plane shapes	Using a *complete* set of logic blocks i.e. squares, oblongs, triangles, circles or similar; red, blue and yellow (or other colours); *large and small*; (possibly thick and thin) to sort attributes, classify, give language of connectives - and, but, or, not, if, then, etc. *Sorting* by decision trees into Venn diagrams. Using rulers for further tessellations, drawing shapes using rulers (and possibly compasses if children can use them) - relating the activities to work in art. Forming equilateral triangles, parallelograms, polygons using paper folding, using string, using drawing.	red <u>and</u> large, red <u>but</u> <u>not</u> large blue <u>or</u> square blue <u>or</u> square <u>but</u> <u>not</u> both <u>if</u> blue <u>then</u> square etc. equilateral, parallelogram, polygon.	e.g. all the logic block pieces.

Topics Concepts Skills	Activities	Language	Recording
8. Symmetry	Looking for planes of symmetry, lines of symmetry using a mirror(s). Symmetrical patterns by folding blots, cutting folded paper, drawing patterns or handwriting or shapes as seen reflected in a mirror. Symmetry of circle about any diameter, reflections as transformations. Radius and area of circle. Symmetry in the environment - cars, faces, leaves, objects in the classroom and home.	reflection, mirror plane of symmetry (3D) line of symmetry (2D) transformations	
9. Similarity (keeping the same shape)	Recognising and making enlargements and smaller models of simple 3D shapes and 2D shapes (using e.g. plasticine, tessellation experiences). Looking at differences (change of size) and similarities (no change of shape) in an object and a model of it. (e.g. a car and a scaled toy and a large scaled map and a smaller scaled map of the same district). Airfix model making. Folding a piece of paper twice to produce a right angle. Making a simple scale plans and models from practical measurements.	shape, angle, fits, looks right, model e.g. "Twice as long and twice as wide and twice as high" right angle	display of constructions tracing the same route on different scaled maps, descriptions in writing, making a solid the same shape only bigger.
10. Further solid shapes	Making solid shapes from plane shapes (nets of solids) e.g.  perhaps 12 regular pentagons into a regular dodecahedron Making solid shapes from frameworks, e.g. straws, Meccano, or similar.	faces, edges, vertices.	

Topics Concepts Skills	Activities	Language	Recording
11. Vertical horizontal, parallel	Describing, making, recognising straight lines and plane (flat) surfaces which are horizontal, vertical, parallel. (Test on hidden half-filled bottle held upright, slanting, upside down, on its side).	upright, level, horizon, vertical, horizontal horizontal and vertical are at right angles. plumb line and water lines are at right angles.	exercises in drawing, e.g. trees on hills, water in bottles, rows of lamp-posts (for parallel lines)
Angles Turns Directions	Recognising right angles in the environment and on unusual or complicated diagrams (e.g. union flag, street plans, wallpapers). Use of spirit level, plumb lines.		"I placed my right angle in the corner - the corner is bigger than a right angle"
Rotations	Deciding whether an angle (as a corner size) is greater than, less than or equal to a right angle. Recognising that a figure can be moved by rotating - using previously made tessellations to find rotations. Recognising directions, N, S, E, W. Given one direction indicating the other three. Finding rotations in the environment (e.g. taps, doors, wheels, etc.) Rotations using a clock face, eight point compass Finding rotational symmetry (e.g. of letter S, propellors, etc).	Direction north, south, east, west, NE, etc. quarter turn, half turn, whole turn.	tessellations showing rotations. "The angle between S and NE is right angles" etc.
12. Translations	Recognising that a figure can be moved by sliding along (without rotation). Find translations in tessellation patterns, wallpapers, and in the general environment.	slide, translation	drawing of plane flying or car driving in straight line showing translation moving clouds and similar.
13. Co-ordinates	Marking a point on a grid given the co-ordinates of the point and vice versa - leading to grid references on Ordinance Survey Maps. Marking several points and joining up to make a sketch. Using squared paper to make enlargements or reductions. Recognising a movement from one point to another point as a translation (intuitive idea of a vector) Properties of figures such as oblongs, squares, rectangles, circles, cube, spheres, etc.	origin, axis, (axes) "along then up"	

APPENDIX FOUR

Children's Interpretations of Line Drawings

Raw data from pilot class

Name	Age when tested	Sex	Ability Band	A1 1 2 3 4 5 6	A2 1 2 3 4	A5 1 2 3 4 5	A6 1 2 3 4 5	A 10 11	A14 1 2 A15 1 2 3 4	A16 1 2	A20 1 2	A23 1 2	27	A28	A30
Martin	8 10	B	A	✓ ✓ ✓ ✓ 0 ✓	✓ X X ?	✓ X 0 ✓ ?	✓ ✓ ✓ X X	✓	not Scored	X X		✓ ?	✓	✓	X
Darren	8 7	B	A	✓ ✓ ✓ ✓ 0 X	✓ X X ✓	✓ ✓ 0 ✓ ✓	X ✓ ✓ ✓ X	✓		X X		✓ X	✓	X	X
Andrew	8 7	B	A	✓ ✓ ✓ ✓ 0 ✓	✓ X X ✓	✓ ✓ 0 ✓ ?	X ✓ ✓ ✓ ✓	✓		X X		✓ X	✓	X	X
Victoria	9 1	G	A	✓ ✓ ✓ ✓ 0 ✓	✓ X X ✓	✓ X 0 X ✓	X ✓ ✓ ✓ ✓	X		X X		✓ X	X	X	X
Claire	8 8	G	A	✓ ✓ ✓ ✓ 0 ✓	✓ X X ✓	✓ ✓ 0 X ✓	X ✓ X X X	X		X X		✓ X	✓	X	X
Peter	8 7	B	B	✓ ✓ ✓ ✓ 0 ✓	✓ ✓ X ✓	✓ ✓ 0 ✓ ✓	✓ ✓ ✓ ✓ ✓	✓		X X		✓ X	✓	X	X
Neil	8 10	B	B	✓ ✓ ✓ ✓ 0 ✓	✓ X X ✓	✓ ✓ 0 ✓ ?	✓ ✓ ✓ ✓ ✓	✓		X X		✓ X	X	X	X
John	8 10	B	B	✓ ✓ ✓ ✓ 0 ✓	✓ ✓ X X	✓ ✓ 0 X X	✓ ✓ ✓ ✓ ✓	X		X X		✓ X	X	X	X
Jody	8 6	B	B	✓ ✓ ✓ ✓ 0 ?	✓ X X ?	✓ ✓ 0 ✓ ?	X ✓ ✓ ✓ X	X		X X		✓ X	✓	X	X
Steven	8 5	B	B	✓ ✓ ✓ ✓ 0 X	✓ X X ?	✓ ✓ 0 X X	✓ ✓ ✓ X X	X		X X		✓ X	✓	X	X
Gary	8 6	B	B	✓ ✓ ✓ ✓ 0 ✓	✓ X X ✓	✓ ✓ 0 X ✓	- - - - -	✓		X X		✓ X	X	X	X
Melanie	8 5	G	B	✓ ✓ ✓ ✓ 0 ?	✓ X X ✓	✓ ✓ 0 ✓ ✓	X ✓ ✓ ✓ X	X		X X		✓ ?	✓	X	X
Teresa	9 1	G	B	X X X X 0 ✓	✓ X X X	✓ ✓ 0 X X	X ✓ ✓ ✓ ✓	✓		X X		✓ X	✓	X	X
Melanie	9 0	G	B	✓ ✓ ✓ ✓ 0 ✓	✓ X X X	✓ ✓ 0 X X	X ✓ ✓ ✓ X	✓		X X		✓ X	X	X	X
Michelle	8 11	G	B	✓ ✓ ✓ ✓ 0 ✓	X X X ✓	X ✓ 0 ✓ ✓	X ✓ ✓ ✓ ✓	X		X X		✓ X	X	X	X
Donna	8 5	G	B	✓ ✓ ✓ ✓ 0 ✓	X X X ✓	✓ ? 0 ✓ X	X ✓ ✓ ✓ ✓	X		X X		✓ X	X	X	X
Darren	8 7	B	C	✓ ✓ ✓ ✓ 0 X	✓ ✓ X ✓	✓ ✓ 0 X X	✓ ✓ ✓ ✓ ✓	X		X X		✓ X	X	X	X
Stephen	8 2	B	C	✓ ✓ ✓ ✓ 0 ✓	✓ X X ✓	✓ ✓ 0 ✓ X	X ✓ ✓ X X	✓		X X		✓ X	X	X	X
Gary	9 1	B	C	✓ ✓ ✓ X 0 X	✓ X X ✓	✓ ✓ 0 X X	X ✓ ✓ ✓ ✓	X		X X		✓ X	✓	X	X
Darren	8 8	B	C	✓ ✓ ✓ ✓ 0 ?	✓ ✓ X ✓	✓ X 0 ✓ ?	X ✓ ✓ ✓ ✓	X		X X		✓ ?	X	X	X
Melanie	8 8	G	C	✓ X ✓ X 0 ✓	✓ X X ✓	✓ ✓ 0 ✓ X	X ✓ ✓ X X	X		X X		✓ X	✓	X	X
Sarah	8 9	G	C	X X X X 0 X	✓ X X ✓	✓ ✓ 0 X ?	X ✓ ✓ ✓ ✓	X		X X		✓ X	X	X	X
Sharon	9 1	G	C	✓ ✓ ✓ X 0 X	✓ X X ✓	✓ ✓ 0 ✓ ✓	X ✓ ✓ ✓ ✓	X		X X		✓ X	✓	X	X
Rachael	8 11	G	C	✓ ✓ ✓ ✓ 0 ✓	✓ X X ✓	✓ ✓ 0 ✓ ?	X ✓ ✓ ✓ ✓	X		X X		✓ X	✓	X	X
Nicholas	8 10	B	D	✓ ✓ ✓ ✓ 0 X	✓ X X ✓	✓ ✓ 0 X X	X ✓ X ✓ ✓	X		X X		✓ X	X	X	X
George	8 2	B	D	✓ ✓ X X 0 X	✓ X X ?	✓ ✓ 0 X ?	X ✓ ✓ ✓ ✓	✓		X X		✓ X	✓	X	X
Melanie	8 4	G	D	✓ X X ✓ 0 X	✓ X X X	✓ ✓ 0 X X	X ✓ ✓ X X	✓		X X		✓ X	✓	X	X
Trina	8 5	G	D	✓ X X ✓ 0 X	✓ X X X	X ✓ 0 X X	X ✓ ✓ ✓ X	✓		X X		✓ X	X	X	X
Natalie	8 4	G	D	✓ X X X 0 X	✓ X X X	✓ ✓ 0 ✓ ?	X ✓ ✓ ✓ X	X		X X		✓ X	X	X	X
Jimmy	8 5	B	D	✓ ✓ ✓ ✓ 0 ✓	✓ X X ✓	✓ ✓ 0 X X	✓ X X X X	X		X X		✓ ✓	X	X	X
Total number of ✓s.				29 28 24 23 - 16	29 4 0 19	30 27 - 14 8	7 28 26 21 15	12		0 0		31 1	14	1	0

✓ indicates a correct response ? indicates unclassified response

x " an incorrect response 0 " response not scored

APPENDIX FIVE

Children's Projective Preferences

Raw Data from Pilot Class

Letter rep. Transformation		(a) Oblique Projective						(b) Affine						(c) Similar						(d) Reflective						(e) Rotational						(f) Piecewise Congruent						
Figure.		P	Q	R	S	T	U	Q	P	T	U	R	S	R	U	P	T	S	Q	S	T	U	P	Q	R	T	S	Q	R	U	P	U	R	S	Q	P	T	
S.P.E left or right		L	R	L	R	R	L	L	R	R	R	L	R	R	L	L	L	R	L	R	L	L	L	L	R	L	L	R	L	L	R	L	R	R	L	R	L	
Item No.		3	9	15	21	29	35	4	10	16	22	30	36	5	11	17	23	31	37	6	12	18	24	32	38	7	13	19	27	33	39	8	14	20	28	34	40	
No. selecting S.P.E.		26	20	27	23	25	26	12	11	13	20	20	20	19	25	23	14	23	21	17	26	26	24	22	25	23	26	21	25	26	23	18	24	24	14	25	25	Total
Martin	1		x		x			x	x				x	x			x		x	x			x	x		x	x	x			x	x			x	x		18
Darren	2											x					x															x						33
Andrew	3							x	x	x							x															x						31
Victoria	4							x				x																				x						33
Claire	5							x							x	x								x											x			31
Peter	6							x									x																		x			33
Neil	7					x			x			x	x																									32
John	8							x									x																		x			33
Jody	9									x								x	x	x	x		x	x	x			x	x	x	x				x		x	22
Steven	10		x				x	x	x			x					x			x						x						x			x			26
Teresa	11															x	x																					34
Melanie	12								x	x	x				x					x									x			x			x			28
Michelle	13								x	x				x			x	x	x													x			x			28
Darren	14		x		x	x			x	x	x		x	x				x		x					x			x			x		x	x				21
Stephen	15		x		x			x	x	x	x		x	x												x		x			x	x			x	x		22
Gary	16							x	x	x				x				x		x														x				29
Darren	17		x					x	x	x	x	x								x												x						28
Melanie	18									x						x							x									x			x			31
Sarah	19															x	x		x																x			32
Sharon	20		x		x			x	x	x	x		x	x						x				x				x					x	x				23
Rachel	21							x	x			x					x																x					31
Nicholas	22	x						x	x	x	x		x													x												29
George	23							x									x		x									x									x	31
Melanie	24													x			x		x													x			x			31
Trina	25								x	x										x																		33
Natalie	26		x						x	x		x		x			x					x		x											x			27
Jimmy	27							x	x	x	x		x							x																		30

APPENDIX SIX

Summative Evaluation : Pretest and Post-test Results

Raw Data for following classes

(a). Pretest for pilot class (Class B)

(b) Pretest for experimental class (Class S)

(c) Post-test for experimental class (Class S   Class L)

(d) Pretest for control class (Class T)

(e) Post-test for control class (Class T   Class M)

Pretest : Pilot class (Class B)

Item	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
	house	table	car	orange	windmill	in squares	in circles	-	-	-	-	(i)	(ii)	(iii)	(iv)	tallest	shortest	cup	T.V.	windows	street	lamp-posts	runway	buildings	-	plan	elevation
Child No.																											
1	9	T	8	T	T	T	T	9	6	6	T	3	T	T	T	7	T	9	9	5	8	7	5	0	6	T	6
2	9	9	T	T	T	T	T	2	6	6	4	3	7	7	3	7	T	7	5	5	8	T	5	4	3	4	0
3	9	9	T	T	T	T	2	T	4	8	T	3	7	7	7	7	T	T	T	3	5	4	5	0	4	T	4
4	5	9	T	T	T	T	T	T	6	T	T	3	0	7	6	7	T	9	T	3	8	4	T	0	1	T	8
5	T	9	T	T	T	T	T	T	6	T	T	3	7	7	4	7	T	7	T	5	5	5	9	5	9	T	6
6	9	9	T	T	T	T	T	7	6	8	T	3	7	7	7	7	T	7	T	5	8	2	T	0	1	T	7
7	9	9	T	4	T	8	T	7	7	T	T	3	7	7	3	7	T	6	0	4	5	2	5	2	4	8	7
8	9	0	T	6	T	8	0	0	6	8	8	3	7	7	3	7	T	7	4	3	6	2	5	1	3	8	4
9	9	9	T	T	T	T	T	6	6	8	8	3	7	7	7	7	T	9	T	3	8	5	8	5	3	8	9
10	9	9	T	T	T	T	8	8	5	6	T	3	7	7	7	7	T	9	T	5	5	4	5	2	1	T	9
11	5	9	8	T	T	T	8	9	7	T	T	3	7	7	T	7	T	7	T	5	1	4	5	2	2	T	1
12	9	9	T	T	T	2	8	8	7	8	T	3	7	7	T	7	T	9	T	5	6	4	5	3	2	T	6
13	9	9	T	T	T	T	T	8	5	8	T	3	7	7	7	7	T	T	T	5	T	8	5	4	3	T	T
14	9	9	T	T	T	T	T	8	3	T	8	3	7	7	7	7	T	9	T	5	5	4	5	2	3	8	6
15	5	T	T	T	T	T	T	9	8	8	8	T	T	7	T	7	T	T	T	8	8	7	5	2	3	T	4
16	9	T	T	T	T	T	T	T	9	8	4	T	T	7	T	7	T	9	T	8	5	5	5	2	3	T	8
17	9	9	T	T	T	T	T	T	9	8	8	T	T	T	T	6	6	T	T	8	8	4	8	2	3	T	9
18	9	9	8	T	T	T	T	8	7	8	8	3	T	7	T	7	T	9	T	5	5	5	5	3	3	T	3
19	9	9	T	T	T	T	T	9	6	6	8	T	7	T	T	7	T	9	T	5	5	7	9	3	4	T	9
20	9	9	T	4	T	T	T	T	7	T	T	7	T	7	T	7	T	9	T	4	8	5	5	2	6	8	T
21	9	T	T	T	2	T	T	9	6	8	T	3	7	7	T	7	T	7	T	5	5	2	5	1	4	T	7
22	9	9	T	4	T	T	T	9	7	8	T	3	T	T	T	7	T	7	T	5	8	7	5	3	3	8	5
23	5	9	T	2	T	T	T	2	2	8	4	3	7	7	T	7	T	T	T	5	5	4	1	0	3	T	7
24	9	9	8	T	T	T	T	0	6	8	4	3	7	7	T	7	T	6	T	5	5	7	5	2	8	T	8
25	T	9	8	2	T	T	2	9	7	9	8	7	7	7	T	7	T	2	T	8	T	7	6	0	3	0	4
26	2	T	T	T	T	T	T	0	7	4	4	3	7	7	3	7	T	7	9	5	8	5	5	0	3	T	7
27	2	T	T	T	T	T	T	8	5	6	8	3	7	7	3	7	T	1	9	5	8	2	5	1	3	T	7
28	2	T	T	T	T	T	8	6	5	4	8	3	7	7	3	7	T	7	T	4	5	2	5	0	4	T	7
29	T	9	T	2	T	T	T	8	8	8	8	3	7	7	0	7	T	7	T	5	5	1	5	1	3	0	7
n	29																										
Mean $\bar{x}$	7.8	8.9	9.7	8.4	9.7	9.6	8.8	7.2	6.1	8.2	8.2	4.2	7.4	7.1	7.1	7.0	9.9	8.1	9.1	5.0	6.4	4.9	5.7	1.8	3.5	8.9	6.3
S.D. σ	2.5	1.7	0.8	2.4	1.5	1.5	2.7	3.1	1.5	2.1	2.1	2.5	1.8	0.6	3.1	0.0	0.7	1.7	2.2	1.4	1.9	2.1	1.9	1.5	1.0	2.2	2.5

Pretest : Experimental Class (Class S)

Item	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
	house	table	car	orange	windmill	in squares	in circles	-	-	-	-	(i)	(ii)	(iii)	(iv)	tallest	shortest	cup	T.V.	windows	street	lamp-posts	runway	buildings	-	plan	elevation
Child No.																											
31	T	9	T	T	T	T	T	1	8	T	T	3	7	7	7	7	T	1	3	4	5	4	5	2	4	8	9
32	T	9	T	6	T	T	T	8	7	8	8	7	T	7	T	3	4	9	9	4	6	2	5	2	3	6	5
33	9	9	T	T	T	T	T	6	8	T	8	3	T	3	T	6	6	9	9	5	5	4	7	3	3	T	8
34	9	9	T	T	T	T	T	7	6	8	8	7	7	7	4	T	T	9	9	9	6	5	9	4	9	T	8
35	9	9	T	T	T	T	T	T	7	T	8	3	7	7	7	7	T	9	6	4	5	4	5	2	4	8	5
36	9	9	T	6	T	T	T	T	4	8	T	3	T	7	T	T	5	9	5	5	6	7	6	2	3	8	T
37	9	9	T	T	T	T	T	T	7	8	T	7	7	7	7	7	T	9	9	5	6	4	5	2	4	T	6
39	9	T	T	T	T	T	T	T	7	8	T	3	7	7	7	7	T	1	3	5	5	4	5	2	3	8	8
41	9	9	T	T	T	T	9	T	9	8	T	7	7	7	T	T	5	9	7	5	5	4	5	1	3	8	4
42	9	9	T	T	T	T	0	8	7	8	8	T	7	7	7	7	T	7	3	4	6	4	5	1	3	8	3
43	T	9	T	T	T	T	T	T	9	T	8	3	T	7	3	T	T	9	9	4	6	4	5	2	3	T	8
44	9	T	8	T	T	T	T	1	6	8	8	3	7	7	3	7	T	1	2	3	5	4	5	1	4	T	7
45	9	9	T	T	T	T	T	T	6	T	8	3	T	7	3	7	T	1	7	5	8	4	6	3	4	T	8
46	T	9	T	T	T	9	T	T	T	T	8	3	T	7	3	T	T	T	7	6	T	T	7	2	6	T	8
47	T	9	T	T	T	8	8	9	7	T	8	3	T	7	3	7	T	1	4	5	3	4	6	1	3	8	8
48	T	T	T	T	T	8	8	T	6	8	8	3	T	7	3	7	T	1	3	4	5	4	1	0	4	4	8
52	9	9	T	4	7	8	T	T	4	8	T	3	T	7	3	3	5	9	7	5	6	7	5	2	3	8	8
55	T	T	T	T	T	T	T	T	6	8	T	3	T	7	T	7	T	9	T	5	3	4	5	1	3	T	8
56	9	9	T	T	T	T	T	T	8	8	8	T	T	7	T	6	6	9	7	5	8	4	5	1	3	T	T
57	9	9	T	4	T	T	T	T	8	T	T	3	7	7	7	6	6	9	T	3	5	4	5	2	4	8	6
61	9	9	T	T	T	T	T	T	5	4	T	3	7	7	3	7	T	8	7	5	8	7	5	2	3	T	3
n	21																										
$\bar{x}$	9·4	9·1	9·9	9·4	T	9·6	9·3	8·2	6·9	8·6	8·8	4·4	8·6	6·8	5·9	7·0	8·5	6·4	6·3	4·7	5·7	4·6	5·5	1·7	3·7	8·6	7·1
s_d	0·5	0·4	0·4	1·6	0·	0·7	2·1	3·0	1·5	1·4	1·0	2·3	1·5	0·8	2·8	1·8	2·2	3·7	2·6	1·2	1·7	1·6	1·4	0·8	1·4	1·5	2·0

Post-test : Experimental Class (Class 2)

Item	1 house	2 table	3 car	4 orange	5 windmill	6 in squares	7 in circles	8 '	9 '	10 '	11 '	12 (i)	13 (ii)	14 (iii)	15 (iv)	16 tallest	17 shortest	18 cup	19 T.V.	20 windows	21 street	22 lamp-posts	23 runway	24 buildings	25 -	26 plan	27 elevation
fig.																											
Child No.																											
31												T	T	T	T	T	5	T	T	9	T	0	9	2	5	T	8
32												T	7	T	T	3	5	9	T	5	5	4	8	1	3	T	8
33												T	T	T	T	T	T	9	T	8	3	0	8	2	8	T	T
34												T	T	T	T	7	T	T	T	8	9	8	T	4	T	T	8
35												3	T	7	T	7	T	9	7	4	8	5	T	2	2	8	7
36												T	T	T	T	3	4	9	9	4	5	4	6	2	3	T	T
37												7	T	7	0	3	4	T	T	8	T	T	9	0	8	T	8
39												3	T	7	T	7	T	9	7	5	9	8	T	1	5	8	7
41												T	T	7	T	3	5	T	9	5	T	9	T	2	4	T	T
42												T	T	T	T	T	T	9	9	6	8	0	T	0	4	8	T
43												T	T	T	T	7	T	9	9	8	9	9	7	2	2	T	T
44												3	7	7	7	T	T	9	7	8	8	8	9	1	2	4	6
45												T	T	0	4	T	5	9	T	8	T	7	T	1	3	T	T
46												T	7	7	0	T	5	T	T	9	T	7	T	1	4	T	8
47												T	T	7	T	T	5	9	9	5	8	9	T	0	3	T	T
48												T	T	7	T	7	T	9	9	8	5	7	T	2	5	T	T
52												T	T	0	T	6	6	9	7	8	9	T	T	1	2	T	8
55												T	7	7	T	6	6	T	7	5	8	4	9	2	2	4	6
56												T	T	T	T	6	6	T	T	9	9	T	9	1	5	T	T
57												T	7	7	0	7	T	9	T	9	9	T	T	0	1	8	7
61												T	T	T	T	7	T	T	9	9	9	2	T	2	5	T	8
n	21																										
Mean $\bar{x}$												9.0	9.4	7.6	8.1	7.1	7.4	9.4	9.0	7.0	8.1	6.2	9.2	1.4	4.1	9.0	8.5
SD σ												2.5	1.2	2.9	1.6	2.5	2.5	0.4	1.2	1.8	2.0	3.4	1.1	0.9	2.2	1.9	1.4

Pretest : Control class (Class T)

Item	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
	house	table	car	orange	windmill	in squares	in circles	'	'	'	'	(i)	(ii)	(iii)	(iv)	tallest	shortest	cup	T.V.	windows	street	lamp-posts	runway	buildings	'	plan	elevation
Child No.																											
63	T	9	T	6	T	T	T	T	8	T	8	3	T	7	T	7	T	9	T	5	9	7	8	2	4	T	T
67	5	9	T	T	T	T	T	T	7	8	T	3	T	7	T	7	8	T	T	4	5	4	5	0	3	T	6
69	9	9	9	T	T	T	T	T	9	8	8	T	T	T	T	3	5	9	7	3	1	4	5	1	1	T	8
70	T	9	T	T	T	T	T	T	5	8	8	T	T	T	T	7	T	9	T	5	5	4	5	1	5	8	4
71	T	5	4	4	T	T	T	T	5	8	T	T	T	T	T	7	6	T	T	4	3	8	5	2	3	T	T
75	9	9	T	T	T	T	T	1	7	8	T	3	7	7	3	7	T	9	8	4	5	4	5	1	3	8	2
76	2	9	T	T	T	T	T	T	4	8	8	3	T	7	3	7	4	9	8	4	5	4	5	2	3	6	2
77	T	9	T	6	T	T	T	T	9	T	8	T	T	T	T	3	5	9	7	4	T	5	8	2	4	8	7
78	T	9	T	T	T	T	T	T	7	T	8	3	T	7	7	3	4	9	T	3	8	7	5	1	4	T	T
80	5	9	4	T	T	T	T	T	9	8	6	3	T	7	3	7	T	T	9	4	6	4	5	2	1	T	6
81	9	T	T	T	T	T	T	T	6	T	8	3	7	7	7	7	T	9	T	5	6	4	8	3	5	T	T
82	2	9	T	4	T	T	T	8	6	T	8	3	7	7	7	7	T	9	9	4	6	7	5	1	5	T	T
83	9	T	T	T	T	T	T	T	8	6	6	3	T	7	7	T	5	T	T	4	6	4	5	1	3	T	3
84	9	9	T	T	T	T	T	9	5	T	8	3	T	7	7	7	T	T	9	4	5	T	5	2	3	T	9
86	9	T	8	T	T	T	T	9	6	8	8	T	T	T	T	7	T	9	T	4	7	4	5	2	4	T	T
89	2	9	T	T	T	T	T	9	6	8	8	3	7	7	7	7	T	9	9	4	3	4	7	1	3	T	5
n	16																										
Mean $\bar{x}$	7.7	8.9	9.2	9.0	T	T	9.4	9.1	6.5	8.6	8.1	5.5	9.4	8.1	8.1	6.2	8.2	8.8	8.9	4.0	6.0	4.6	6.0	1.5	3.5	9.4	7.0
SD σ	2.9	0.9	1.8	2.0	0	0	1.6	2.1	1.5	1.3	1.0	3.3	1.2	1.4	2.5	1.6	2.5	1.5	1.7	0.7	2.0	1.2	1.3	0.8	1.1	1.1	3.0

Post-test : Control class (Class M)

Item	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
	house	table	car	orange	windmill	in squares	in circles	'	'	'	'	'	(i)	(ii)	(iii)	(iv)	tallest	shortest	cup	T.V.	windows	street	lamp-posts	runway	buildings	'	plan	elevation
Child No.																											
63												T	T	7	T	9	T	T	9	9	8	T	T	2	9	T	8
67												T	T	7	T	7	T	T	9	3	5	4	T	1	2	T	T
69												T	T	7	T	9	T	T	9	5	8	7	T	2	2	T	8
70												T	T	T	T	7	T	9	9	5	6	5	8	2	4	T	4
71												3	T	T	T	3	4	T	9	8	8	4	T	3	4	T	7
75												T	7	T	T	7	T	9	T	4	5	4	5	2	3	2	8
76												T	T	T	T	T	T	9	9	5	5	4	6	2	1	T	0
77												T	T	T	T	7	T	T	9	5	5	7	T	2	5	T	8
78												T	T	T	T	T	5	7	7	5	8	2	6	2	4	T	8
80												T	T	7	T	3	5	7	9	5	T	4	T	1	3	T	8
81												T	T	7	T	9	0	T	T	5	9	7	T	2	3	T	6
82												T	T	7	T	7	T	T	9	8	8	9	9	0	2	T	8
83												0	T	7	T	T	5	7	7	8	6	4	9	1	3	T	8
84												T	T	T	T	7	T	7	9	4	6	0	1	2	2	T	T
86												T	T	7	T	0	T	9	T	8	5	4	T	2	3	T	T
89												T	T	T	4	7	T	9	9	5	8	2	T	0	4	T	8
n	16																										
Mean $\bar{x}$												8.	9.8	8.5	9.6	7.0	8.1	9.1	8.9	5.8	6.7	4.8	8.4	1.6	3.4	9.5	7.4
s.d. σ												2.9	0.1	1.	1.5	2.7	3.1	1.1	0.8	1.8	1.6	2.5	2.5	0.7	1.8	1.9	2.4

APPENDIX SEVEN

Some Strategy Games Using Desargues' Theorem

Published articles

(a) Mathematics in School January 1982, 11 (1), 30

(b) The Mathematics Teacher November 1982, 75 (8), 652-3

Some Strategy Games Using Desargues's Theorem

by **A. J. Salisbury,** School of Education
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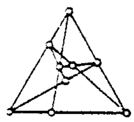
Stephen Krulik (1977) advocates the use of games in the classroom as they "can transform the classroom from an assemblage of passive spectators into a vibrant laboratory of active participants". He suggests that the game of Noughts and Crosses contains the sort of strategies which may be appropriate for students. He further suggests that new strategies can evolve when the shape of the board is modified.

A Krulik Noughts and Crosses board contains nine cells in which the players may place Xs or Os in an effort to get three in a row (Figure 1).

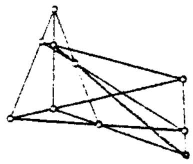
Fig. 1



This board has several advantages over the usual 3 × 3 square arrangement. Each of the nine cells lies on exactly three rows and each of the nine rows contain exactly three cells.

A similar Noughts and Crosses game may be played on the usual Desargues's diagram (Figure 2).

Fig. 2



This is a 10-3, 3-10 figure; each of the 10 cells lies on exactly three rows and each of the 10 rows contain exactly three cells. Because of the projective nature of the diagram all the cells are of equal importance and no cell is especially strategic for the first move. It is perhaps unfortunate, however, that the diagram does not have the pleasing symmetry of the Krulik figure. Because of the extra cell is becomes more likely for one or both competitors to obtain a second row in the same game. This may be rewarded by a bonus such as three extra points for the completion of a second row compared with one point for a first row. One game on the board, invented without any prompting by a 10-year-old boy involved the use of four counters each.

Each player uses a counter in turn until eight cells are occupied. Then all eight counters stay on the board and each player in turn moves one of his counters to an adjacent vacant cell (without jumps) in an effort to complete a row or prevent his opponent from completing one. Needless to say the strategy for this game is very different. Some limit needs to be placed on the length of each game.

It is well known that Desargues's Theorem is self dual. That is, that each true statement made about rows and cells remain true when the words row and cell are interchanged. So for example as each row has three cells, each cell has three rows passing through it.

A dual Noughts and Crosses game may be played. This involves using thin transparent coloured strips. Each player in turn places a strip of his colour along one of the rows and scores one point if he obtains three strips through one cell, three extra points if he manages to do this twice in the same game. Both the Krulik 9-3, 3-9 figure and the Desargues 10-3, 3-10 figure makes interesting variations of Noughts and Crosses with distinctive strategies — the dual game is quite difficult and has some potential for older children.

Reference
Krulik, S. (1977) "Problems, Problem Solving, and Strategy Games", *Mathematics Teacher*, 70, 649-652.

SOME STRATEGY GAMES USING DESARGUES'S THEOREM

By ANDREW JOHN SALISBURY
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The use of games in the classroom can "transform the classroom from an assemblage of passive spectators into a vibrant laboratory of active participants" (Krulik 1977). One technique that I have employed is to modify the shape of the pattern used for tic-tac toe and then to offer students opportunities to develop new strategies. Krulik used a board containing nine cells in which the players attempt to place three Xs or three Os in a row (fig. 1).

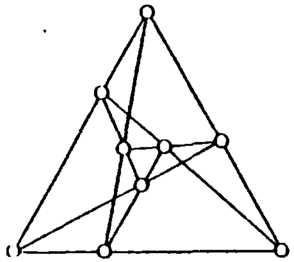

Fig. 1

This board has several advantages over the usual three-square-by-three-square arrangement. Each of the nine cells lies on exactly three rows, and each of the nine rows contains exactly three cells. The usual tic-tac-toe arrangement has the middle cell on four rows, the corner cells each on three rows, and the others on two rows, which is a less satisfactory arrangement. The Krulik board has nine winning combinations instead of the usual eight on the standard board.

A Desargues Configuration

I began by using a board employing a Desargues configuration with ten cells; each cell is the intersection of exactly three rows, and each row has exactly three cells on it (fig. 2). Because of the projective nature of figure 2, all the cells are of equal importance, and no cell is especially strategic for the first move.

The students initially played the usual tic-tac-toe game with a goal of obtaining three identical marks in a row. Instead of marking the board with the usual Os and Xs, it is helpful to use two sets of colored counters. The board can then be used again. A good description of the configuration and its origin is found in Kline (1973).

I found that a more satisfying game with

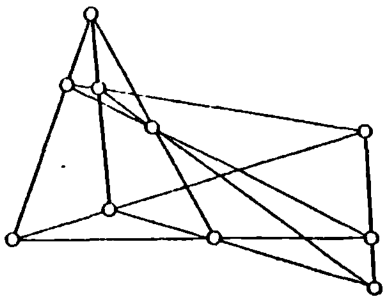

Fig. 2

Desargues's configuration is to continue the game until all the cells are occupied. The winner is the person to complete the most rows. It is difficult on occasions to obtain a row without allowing an opponent to obtain one in return. In addition, the extra cell makes it more likely for one or both competitors to obtain a second row in the same game.

From observing students playing this game, I developed a scoring system giving points for completing rows—one point for a first row and an additional three points for a second (or third) row in the same game. This structure helps to ensure that the players continually look ahead.

Another activity on the same board was invented without any prompting by my student, Steven Cowles. Each player begins with four counters. The players take turns placing their counters until eight cells are occupied. All the eight counters stay on the board, and each player in turn moves one counter, without jumping another, to an adjacent vacant cell in an effort to complete a row or to prevent an opponent from completing one. Needless to say, this activity requires a different strategy, and some time limit or scoring limit needs to be placed on the length of each game. "The first one to get more than five rows is the winner" was found to be a suitable restriction.

Dual Variations

Desargues's configuration is self-dual; that is, each true statement made about rows and cells remains true when the words *row* and *cell* are interchanged. For example, as there are ten cells, there are also ten rows, and since each row has three cells (on it), each cell has three rows (through it).

An activity capitalizing on this duality is a dual tic-tac-toe (*tac-tic*-toe?) game played with thin, transparent colored strips. Each player begins with five strips of a given color. The winner is the first player to have three strips through a cell. My students and I further refined the activity by changing the scoring system, as before, giving three extra points if a player manages to do this twice in the same game.

The dual games with colored strips appear to be more difficult than the corresponding games played with counters. I found that some competitors playing against better strategists tended to lose interest after a string of successive defeats, but by rearranging the players or joining in myself, it was possible to prevent this from happening too often.

I thought it valuable for students to be involved in spatial and geometrical games, and they enjoyed these activities.

REFERENCES

Kline, Morris. *Mathematical Thought from Ancient to Modern Times*. New York: Oxford University Press, 1973.

Krulik, Stephen. "Problems, Problem Solving, and Strategy Games." *Mathematics Teacher* 70 (November 1977):649-52.